Quiz 1 Solutions

P1. Compute $\int_{1/e}^{e} \frac{\ln x}{\sqrt{x}}\,dx$.

$u = \ln x \qquad dv = \frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{\sqrt{x}}{1/2} = 2\sqrt{x}$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int \frac{2\sqrt{x}}{x}\,dx$$
$$= 2\sqrt{x}\ln x - 2\int \frac{dx}{\sqrt{x}}$$
$$= 2\sqrt{x}\ln x - 4\sqrt{x}$$

So, $\int_{1/e}^{e} \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x}\Big|_{1/e}^{e} = 2\sqrt{e}\ln e - 4\sqrt{e} - \left(\frac{2}{\sqrt{e}}\ln\left(\frac{1}{e}\right) - \frac{4}{\sqrt{e}}\right)$

$$= 2\sqrt{e} - 4\sqrt{e} - \left(-\frac{2}{\sqrt{e}} - \frac{4}{\sqrt{e}}\right)$$
$$= -2\sqrt{e} + \frac{6}{\sqrt{e}}$$

P2. $R(t) = \frac{1}{2}t^2$ **want:** $e^{rT}\int_0^T \frac{1}{2}t^2 e^{-rt}\,dt = \frac{e^{rT}}{2}\int_0^T t^2 e^{-rt}\,dt$

first we compute $\int t^2 e^{-rt}\,dt$.

$u = t^2 \qquad dv = e^{-rt}\,dt$

$du = 2t\,dt \qquad v = -\frac{e^{-rt}}{r}$

so $\int t^2 e^{-rt}\,dt = -\frac{t^2 e^{-rt}}{r} - \int 2t\left(-\frac{e^{-rt}}{r}\right)dt$

$$= -\frac{t^2 e^{-rt}}{r} + \frac{2}{r}\int t\,e^{-rt}\,dt$$

$u = t \qquad dv = e^{-rt}\,dt$

$du = dt \qquad v = -\frac{e^{-rt}}{r}$

$$\int t\,e^{-rt}\,dt = -\frac{t\,e^{-rt}}{r} - \int -\frac{e^{-rt}}{r}\,dt$$

$$= -\frac{t^2 e^{-rt}}{r} + \frac{2}{r}\left(-\frac{t\,e^{-rt}}{r} + \frac{1}{r}\int e^{-rt}\,dt\right)$$
$$= -\frac{t^2 e^{-rt}}{r} + \frac{2}{r}\left(-\frac{t\,e^{-rt}}{r} - \frac{1}{r^2}e^{-rt}\right)$$
$$= -\frac{t^2 e^{-rt}}{r} - \frac{2te^{-rt}}{r^2} - \frac{2e^{-rt}}{r^3}$$

thus, $\frac{e^{rT}}{2}\int_0^T t^2 e^{-rt}\,dt = \frac{e^{rT}}{2}\left(-\frac{t^2 e^{-rt}}{r} - \frac{2te^{-rt}}{r^2} - \frac{2e^{-rt}}{r^3}\right)\Big|_0^T$

$$= \frac{e^{rT}}{2}\left(-\frac{T^2 e^{-rT}}{r} - \frac{2Te^{-rT}}{r^2} - \frac{2e^{-rT}}{r^3} - \left(-0 - 0 - \frac{2}{r^3}\right)\right)$$

$$= \frac{e^{rT}}{2}\left(-\frac{T^2 e^{-rT}}{r} - \frac{2Te^{-rT}}{r^2} - \frac{2e^{-rT}}{r^3} + \frac{2}{r^3}\right)$$

$$= -\frac{T^2}{2r} - \frac{T}{r^2} + \frac{e^{rT}-1}{r^3} \leftarrow$$

either form is fine

$$= \frac{-T^2r^2 - 2Tr + 2e^{rT} - 2}{r^3} \leftarrow$$