Lecture Notes from Wednesday, June 29th, 2014 (MAC2234)

We generalize the logistic equation to a variable population size: $\frac{dy}{dt} = ky\left(1 - \frac{y}{N(t)}\right)$

To solve this equation we make the substitution $v = 1/y$. This gives us $\frac{dv}{dy} = -\frac{1}{y^2} = -v^2$,

and thus $$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = -v^2 ky\left(1 - \frac{y}{N(t)}\right) = -v^2 \frac{k}{v}\left(1 - \frac{1}{vN(t)}\right)$$

$$\longrightarrow \quad \frac{dv}{dt} = -kv + \frac{k}{N(t)}$$

This is a first order linear differential equation in v that we can solve using the method of the integrating factors.

$$\frac{dv}{dt} + kv = \frac{k}{N(t)} \qquad \mu(t) = \int k\,dt = kt \quad \rightarrow I(t) = e^{kt}$$

$$e^{kt}v(t) = \int \frac{ke^{kt}}{N(t)}\,dt$$

$$v(t) = e^{-kt}\int \frac{ke^{kt}}{N(t)}\,dt \qquad \text{So} \qquad y(t) = \frac{e^{kt}}{\int \frac{ke^{kt}}{N(t)}dt}$$

· Ex. Logistically growing subset of an exponentially growing population:

In this case, $N(t) = N_0 e^{rt}$. So $y(t) = \frac{e^{kt}}{\int \frac{ke^{kt}}{e^{rt}}dt} = \frac{e^{kt}}{\int ke^{(k-r)t}dt} = \frac{e^{kt}}{\frac{k}{(k-r)}e^{(k-r)t} + C}$

$$= \frac{k-r}{ke^{-rt} + Ce^{-kt}}$$

Exercise: Derive an equation modeling a logistically growing subset of a logistically growing population.