Midterm Exam 2

MAC2234: Survey of Calculus II   Solutions   Tuesday, July 15th, 2014.

Score			
P1	5 / 5		
P2	5 / 5	**P5**	5 / 5
P3	5 / 5	**P6**	5 / 5
P4	5 / 5	**P7**	5 / 5
		EX2	35 / 35

Question 1. *(Techniques of Single Variable Integration)*
Evaluate

$$\int_e^{e^2} \frac{\ln(\ln x)\ln x}{x}\,dx$$

Let $u = \ln x$, so that $du = \frac{dx}{x}$.

Then $\displaystyle\int_e^{e^2} \frac{\ln(\ln x)\ln x}{x}\,dx = \int_1^2 u \ln u \, du$

Integrating by parts using $w = \ln u$, $dv = u\,du$, $dw = \frac{du}{u}$, $v = \frac{u^2}{2}$, we get

$$\int_1^2 u\ln u\,du = \left.\frac{u^2\ln u}{2}\right|_1^2 - \frac{1}{2}\int_1^2 u\,du$$

$$= 2\ln 2 - \left.\left(\frac{u^2}{4}\right)\right|_1^2$$

$$= 2\ln 2 - \frac{3}{4}$$

$$= \ln 4 - \frac{3}{4}$$

$$\approx 0.636294$$

Let $\bar{f}(x) = e^{-x^2/2}$.

i	x_i	$\bar{f}(x_i)$
0	-2.00	0.135
1	-1.5	0.325
2	-1.0	0.607
3	-0.5	0.882
4	0	1.000
5	0.5	0.882
6	1.0	0.607
7	1.5	0.325
8	2.0	0.135

(note: it's easier to use Simpson's on $\int_{-2}^{2} e^{-x^2/2}\,dx$, then multiply by $\frac{1}{\sqrt{2\pi}}$.)

Using Simpson's rule, we approximate $\int_{-2}^{2} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \bar{f}(x)\,dx$ by

$$\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{0.5}{3}\right)\big(0.135 + 4\cdot 0.325 + 2\cdot 0.607 + 4\cdot 0.882 + 2\cdot 1 + \cdots$$

$$\cdots + 4\,0.882 + 2\cdot 0.607 + 4\cdot 0.325 + 0.135\big)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{0.5}{3}\right)\cdot 14.354$$

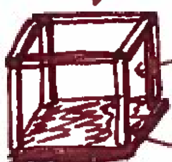
$$= \left(\frac{1}{\sqrt{2\pi}}\right) 2.392$$

$$= 0.954$$

Question 3. *(Multivariable Optimization)*

You're building a custom made yurt shaped like a rectangular prism. The floor of the yurt must be made of oak, which costs \$2 per square foot. The frame (i.e. the remaining edges of the prism) must be made of bamboo poles, which are \$7 per foot. The walls and ceiling of the yurt must be made of canvas tarp, which costs 50 cents per square foot. The customer gives you a lump sum of \$750 and asks you to build the most spacious yurt you can. What dimensions will yield the yurt with the greatest volume?

canvas ceiling: $\frac{1}{2}xy$



canvas walls: $\frac{1}{2}(2xz + 2yz) = xz + yz$

bamboo support frame: $7(4z + 2x + 2y)$

oak floor: $2xy$

cost: $\frac{5}{2}xy + xz + yz + 7(4z + 2x + 2y)$

constraint is cost $= 750$, so $g(x,y,z) = \frac{5}{2}xy + xz + yz + 7(4z + 2x + 2y) - 750$.

We're maximizing volume, so the Lagrangian is

$$\boxed{\mathcal{L}(x,y,z,\lambda) = xyz - \lambda\left(\frac{5}{2}xy + xz + yz + 7(4z + 2x + 2y) - 750\right)}$$

Setting $\mathcal{L}_x, \mathcal{L}_y$ to 0 and setting the λ's equal to each other gives us that $\boxed{x = y}$, as we would expect from the symmetry of the problem.

Replacing y with x in $\mathcal{L}_y$ and equating this λ to the one from $\mathcal{L}_z$ gives us that $\boxed{z = \dfrac{\frac{5}{2}x^2 + 14x}{x + 28}}$.

Taking $g\left(x, x, \dfrac{\frac{5}{2}x^2 + 14x}{x+28}\right) = 0$ gives us $\boxed{\dfrac{7.5x^3 + 196x^2 + 426x - 21000}{x + 28} = 0}$

Numerically, we then have and thus $\boxed{\begin{array}{l} x = y = 8.237 \\ z = 7.864 \end{array}}$ (which, by the way, yields a volume of 534 cubic ft.)

Question 4. *(Total Differentials and Double Integrals)*
Kanye West has decided that when he dies he wishes to be buried in a solid gold, cone-shaped mausoleum, which is to be built in Giza and referred to as "the Greatest Pyramid." He insists that the building be 148 [redacted] meters tall (one meter taller than the Great Pyramid) with a base radius of 231 meters (one meter wider than the base of the Great Pyramid). The contractor he hires measures each of these lengths with a maximum error of five centimeters. Estimate the maximum error in calculating the surface area of Kanye West's tomb.

Note: Because I am a nice guy, here is the formula for the surface area of a cone.

$$SA(r,h) = \pi r\left(r + \sqrt{h^2+r^2}\right)$$

From the problem we have $dr = 0.05$ and $dh = 0.05$.

We need to compute $dSA = SA_r\,dr + SA_h\,dh$.

$$SA_r = \frac{\pi\left(r+\sqrt{h^2+r^2}\right)^2}{\sqrt{h^2+r^2}} \qquad SA_h = \frac{\pi r h}{\sqrt{h^2+r^2}}$$

Using the r and h from the problem, we get

$$SA_r \approx 2924.35 \qquad SA_h \approx 391.496$$

So we have a maximum possible error of

$$\boxed{\begin{aligned} dSA &= (2924.35)(0.05) + (391.446)(0.05) \\ &= 165.792 \end{aligned}}$$

when the contractor calculates the surface area of Kanye West's extravagant mausoleum.

(note: c.f. page 154 of the new book for a similar example.)

Question 5. *(Elementary Differential Equations)*
The Gompertz equation is a population growth model similar to logistic growth. A group of statisticians famously used its solution in the early 2000s to forecast the rapid expansion of the cell phone industry, in a study whose predictions remain highly accurate to this day. The Gomertz equation is also used in medicine to analyze the growth of tumors.

Anyway, here it is:

$$\frac{dy}{dt} = ky\ln\left(\frac{M}{y}\right)$$

Find the general solution to the Gompertz equation and discuss its long term behavior.

$$\int \frac{dy}{y\ln\left(\frac{M}{y}\right)} = \int k\,dt = kt + c$$

$$-\ln\left(\ln\frac{M}{y}\right) = kt + c$$

$$\ln\frac{M}{y} = e^{-kt-c} = ce^{-kt} \qquad (c \overset{by}{\leftrightarrow} e^{-c})$$

$$\frac{M}{y} = e^{ce^{-kt}}$$

$$y = Me^{-ce^{-kt}}$$

Assuming that $k > 0$, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Me^{-ce^{-kt}} = M$.

Therefore, we see that Gompertz growth approaches a constant carrying capacity M in the long term.

(By the way, did anybody notice that $\frac{d^2y}{dt^2} = k\frac{dy}{dt}\left(1 - \log\left(\frac{M}{y}\right)\right)$?
interestingly, looks like logistic eq'n, hmmm...)

Question 6. *(Numerical Differential Equations)*
You've been given the exciting opportunity to work at Geezbook, the new social network for old people that is spreading through Florida like wildfire. Your job is to predict how much money the company will make during the coming year. Here is the information you will need:

- The site has $\overset{S_0}{\underline{32\,596}}$ subscribers when you arrive on the job, and gains $\overset{\varepsilon}{\underline{7900}}$ new subscribers over the course of the following three months.
- According to the 2010 census, there are $\overset{P}{\underline{3\,259\,602}}$ people over the age of 65 living in Florida. (You may assume the number of subscribers is growing logistically within this constant carrying capacity.)
- Each subscriber pays $\overset{F}{\underline{\text{fifteen}}}$ dollars per month for Geezbook's service, which is withdrawn continuously from his or her bank account.
- The CFO reinvests the company's wealth at a continuously compounding interest rate of $\overset{r}{\underline{5.9\%}}$.

Subscribers at time t: $s(t) = \dfrac{P}{1+be^{-kt}}$

By $s(0) = S_0$, we get $\boxed{b = \dfrac{P-S_0}{S_0} \approx 99}$

By $s(3) = S_0 + \varepsilon$, we get $\boxed{k = \dfrac{1}{3}\ln\dfrac{(P-S_0)(S_0+\varepsilon)}{(P-S_0+\varepsilon)S_0} \approx 0.0731549}$

So, $s(t) = \dfrac{3259602}{1+99e^{-0.0731549t}}$.

Now, we have $\boxed{\dfrac{dW}{dt} = rW + Fs(t)}$. To solve this exactly, we would need to be able to integrate $\displaystyle\int \frac{Pe^{-rt}}{1+be^{-kt}}\,dt$, but this is not an elementary integral. So, we have to use Euler's method. Let's choose a step size of one month and find $W(12)$. At $t=0$ we haven't made any money yet, so $W(0)=0$. (SEE ATTACHED TABLE)

The result is $\boxed{\$11,976,059}$

Question 7. *(Probability)*
GE aviation has determined that their airplane turbines have a probability of failure given by

$$f(t) = ae^{-at} \quad , \quad \text{where} \quad a = 0.0588235$$

and t is in years. What is the expected lifespan of an airplane turbine? What is the median lifespan?

The expected lifespan is $\int_{-\infty}^{\infty} tf(t)\,dt = \int_{0}^{\infty} tf(t)\,dt = \int_{0}^{\infty} tae^{-at}\,dt = \lim_{R\to\infty} \left. \frac{-e^{-at}(1+at)}{a} \right|_0^R$

$= \lim_{R\to\infty} \frac{1-e^{-aR}(1+aR)}{a}$

$= \frac{1}{a}$

≈ 17

The median lifespan is m such that $\int_{-\infty}^{m} f(t)\,dt = \frac{1}{2}$

$\int_{0}^{m} ae^{-at}\,dt = \frac{1}{2}$

$\left. -e^{-at} \right|_0^m = \frac{1}{2}$

$1 - e^{-am} = \frac{1}{2}$

$e^{-am} = \frac{1}{2}$

$-am = \ln\frac{1}{2}$

$am = \ln 2$

$m = \frac{\ln 2}{a}$

≈ 11.7835

alternatively, we could use Simpson's rule to numerically approximate $e^{12r}\int_0^{12} \frac{15P}{1+be^{-kt}} e^{-rt}\,dt$, which gives a slightly larger answer (just over 12 million)