Answer Key

Midterm Exam 1

MAC2234: Survey of Calculus II

Thursday, June 12th, 2014.

Score	
P1	5 / 5
P2	5 / 5
P3	5 / 5
P4	5 / 5
EX1	20 / 20

great job, myself!

Question 1. (5 points.)
Evaluate

$$\int_{a-1}^{a+1} (x-a)\ln|x-a|\, dx$$

and justify your answer.

The short and easy way is to notice that this function becomes odd under a change of variables:

$$u = x-a, \quad du = dx \quad \to \quad \int_{-1}^{1} u \ln|u|\, du = 0$$

The longer, more tedious way is to evaluate the definite integral by hand. This can be done by integration by parts, and by splitting the integrand into a piecewise function

$$(x-a)\ln|x-a| = \begin{cases} (x-a)\ln(x-a) : x \geq a \\ (x-a)\ln(a-x) : x < a \end{cases}$$

Then evaluating

$$\int_{a-1}^{a} (x-a)\ln(a-x)\, dx + \int_{a}^{a+1} (x-a)\ln(x-a)\, dx$$

gives you two expressions of opposite sign, so they sum to 0.

(Note: one subtle aspect of this problem (which I didn't grade on) is that the integral is, technically, improper, because $\ln|x-a|$ is not defined at $x=a$. So what we really should be doing is writing

$$\lim_{R \to a} \int_{a-1}^{R} (x-a)\ln|x-a|\, dx + \lim_{S \to a} \int_{S}^{a+1} (x-a)\ln|x-a|\, dx .)$$

To solve this, we exploit $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ to combine Trapezoid & Simpson's rules, and account for the different Δx's.

Question 2. (5 points.)
You are thinking about buying a billboard on the highway to advertise your business, but you're not sure if enough people will see it. Showing great ingenuity, you install a camera on the overpass to measure the flow of traffic. Unfortunately it's kind of a shoddy camera, and it loses all but 14 records, taken at the following times.

differences:

S	29	8:00 pm	2.60
S	29	8:29 pm	8.50
T	28	8:58 pm	28.64
T	32	9:26 pm	32.98
T	26	9:58 pm	25.48
T	16	10:24 pm	170.92
S	22	10:40 pm	340.44
S	22	11:02 pm	354.20
T	22	11:24 pm	67.82
T	49	11:46 am	91.02
T	15	12:35 am	208.66
S	20	12:50 am	273.46
S	20	1:10 am	219.06
		1:30 am	131.34

$\frac{29}{60\cdot 3}(2.6 + 4\cdot 8.5 + 28.64) = 10.51$

$\frac{28}{60\cdot 2}(28.64 + 32.98) = 14.38$

$\frac{32}{60\cdot 2}(32.98 + 25.48) = 15.59$

$\frac{26}{60\cdot 2}(25.48 + 170.92) = 42.56$

$\frac{16}{60\cdot 2}(170.92 + 340.44) = 68.18$

$\frac{22}{60\cdot 3}(340.44 + 4\cdot 354.2 + 67.82) = 22[illegible]01$

$\frac{22}{60\cdot 2}(67.82 + 91.02) = 29.12$

$\frac{49}{60\cdot 2}(91.02 + 208.66) = 122.37$

Approximate the total number of cars that pass by your camera between the hours of 8:00pm and 1:30am.

$\frac{15}{60\cdot 2}(208.66 + 273.46) = 60.27$

$\frac{20}{60\cdot 3}(273.46 + 4\cdot 219.06 + 131.34)$

$= 142.338$

for a grand total of $740.748 \approx \boxed{741 \text{ cars}}$

(or, if you measured by minutes, 4444 cars) without back, ends up being $r=h$, so $r=\sqrt[3]{\frac{4V}{\pi}}=h$, $SA=3\sqrt[3]{2\pi V^2}$, price $=\$70.8$

Question 3. (5 points.)
Toronto Mayor Rob Ford orders a custom polyurethane drum to store hold 8 litres (8000 cubic centimeters) of lysergic acid diethylamide. He specifies that the drum should be open on top, for easy access, and for the base of the drum to be a half-circle, so it can be mounted at waist level on the wall of his office. Assuming that polyurethane costs 3.2¢ per square centimeter, what is the smallest possible cost to manufacture Rob Ford's custom storage tank?

Call this V, for now.

$$f(r,h)=\underbrace{\frac{\pi r^2}{2}}_{\text{bottom}}+\underbrace{2rh}_{\text{back}}+\underbrace{\frac{2\pi rh}{2}}_{\text{outer part}} \qquad g(r,h)=\frac{\pi r^2 h}{2}-V$$

$$\mathcal{L}(r,h,\lambda)=\frac{\pi r^2}{2}+2rh+\frac{2\pi rh}{2}-\lambda\left(\frac{\pi r^2 h}{2}-V\right)$$

$$\mathcal{L}_r(r,h,\lambda)=\pi r+2h+\pi h-\frac{2\pi hr\lambda}{2}=\pi r+2h+\pi h-\pi hr\lambda=0$$

$$\mathcal{L}_h(r,h,\lambda)=2r+\pi r-\lambda\frac{\pi r^2}{2}=0$$

$$\lambda=\frac{\pi r+2h+\pi h}{\pi hr}=\frac{1}{h}+\frac{2}{\pi r}+\frac{1}{r}$$

$$\lambda=\frac{(2r+\pi r)2}{\pi r^2}=\frac{4}{\pi r}+\frac{2}{r}$$

$$\frac{1}{h}+\frac{2}{\pi r}+\frac{1}{r}=\frac{4}{\pi r}+\frac{2}{r}$$

$$\pi r+2h+\pi h=4h+2\pi h$$

$$h=\frac{\pi}{2+\pi}r \quad \text{so} \quad V=\frac{\pi r^2}{2}\left(\frac{\pi}{2+\pi}r\right)$$

$$r=\sqrt[3]{\frac{2(2+\pi)}{\pi^2}V}$$

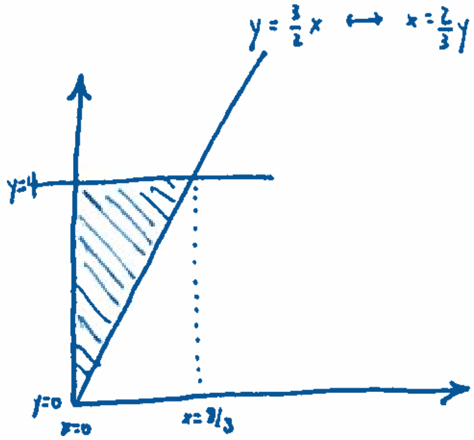
plugging that into f, we get

$$f(r,h)=3\sqrt[3]{\frac{(\pi+2)^2V^2}{2\pi}}\approx 1937.25$$

... times 3.2 cents $=\boxed{\$61.99}$

Question 4. (5 points.)
Compute

$$\int_0^{\frac{8}{3}} \int_{\frac{3x}{2}}^{4} \frac{1}{\sqrt{y^2+9}}\, dy\, dx$$



$$\downarrow$$

$$= \int_0^4 \int_0^{\frac{2}{3}y} \frac{1}{\sqrt{y^2+9}}\, dx\, dy$$

$$= \int_0^4 \left(\frac{x}{\sqrt{y^2+9}} \right) \Bigg|_0^{\frac{2}{3}y} dy$$

$$= \frac{2}{3} \int_0^4 \frac{y}{\sqrt{y^2+9}}\, dy$$

$u = y^2 + 9$

$du = 2y\, dy$

$$= \frac{1}{3} \int \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{3} \left(2\sqrt{u} \right)$$

$$= \frac{1}{3} \left(2\sqrt{y^2+9} \right) \Bigg|_0^4$$

$$= \frac{2}{3} (5 - 3)$$

$$= \frac{4}{3}$$