# Group Theory: Extra Credit Assignment #3 and Exam #3 Review.

Assigned April 18th. Due the day of Exam #3.

This exam is about Sylow theory and its relationship to finite nonabelian simple groups. You must be able to *state, prove, and apply* the class equation and all three Sylow theorems.

Each question followed by a $\star$ may be turned in for 1% extra credit on your total grade by Monday.

**Question 1.**
Show that a group of each of the following orders is not simple.

(a) 6545,

(b) 1365,

(c) 2907,

(d) 132,

(e) 462.

(f) $pqr$, where $p, q, r$ are distinct primes.

(g) $p^2q$, where $p, q$ are distinct primes. $\star$

**Question 2.**
Let $|G| = pq$ for primes $p, q$ with $p < q$. Show that if a Sylow $p$-subgroup of $G$ is normal, then $G$ is cyclic.

**Question 3.**
Prove that if $G$ has order 231 then $Z(G)$ contains a Sylow 11-subgroup of $G$ and a Sylow 7-subgroup is normal in $G$. Then prove that if $G$ has order 385 then a Sylow 11-subgroup is normal in $G$ and $Z(G)$ contains a Sylow 7-subgroup of $G$.

**Question 4.**
State and prove the class equation. Deduce that

(a) $p$-groups have nontrivial centers,

(b) if $[G : Z(G)] = n$, any conjugacy class has at most $n$ elements.

**Question 5.**
Let $p$ be the smallest prime divisor of $|G|$. If $P \in \mathrm{Syl}_p(G)$ is cyclic, prove that $N_G(P) = C_G(P)$. *Hint. Consider* $\theta : N_G(P) \to \mathrm{Aut}(P)$ *by* $\theta(g) = \theta_g$ *where* $\theta_g(x) = g^{-1}xg$. $\star$

**Question 6.**
Let G have order 1575. Prove that if a Sylow 3-subgroup of G is normal then a Sylow 5- and a Sylow 7-subgroup are normal.

**Question 7.**
How many elements of order 7 must there be in a simple group of order 168? Can such a group contain an element of order 21? $\star$