Group Theory: Exam #2 Review.

For the exam this Friday, I expect you to be comfortable with the structure theorem for finitely generated abelian groups, the isomorphism theorems, and the computation of quotient groups. You may expect that the exam questions will be similar to those below, with some modifications.

Abelian Groups.

Remark. The *primary decomposition* of a finite abelian group refers to its complete factorization into a direct sum of cyclic groups of prime power order. The components are conventionally written left to right ordered first by prime, then by exponent. Here are some examples.

Group	Primary Decomposition
$\mathbb{Z}_8$	$\mathbb{Z}_8$
$\mathbb{Z}_{100}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_{25}$
$\mathbb{Z}_{14} \oplus \mathbb{Z}_{36}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$
$\mathbb{Z}_{18} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{22}$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{11}$

Today in class we struggled to find a notation to express the general case. You may find the wording helpful in questions 1 and 3.

Question 1.
Suppose that G is a finitely generated abelian group. Prove that

$$G \cong \mathbb{Z}^m \oplus \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_r}$$

for some $r, m \in \mathbb{N}$, where the q_i are powers of (not necessarily distinct) primes.

Question 2.
Classify the abelian groups of each of the following orders up to isomorphism. Provide the number of distinct isomorphism classes along with their primary decompositions.

(a) $1800 = 2^3 3^2 5^2$,

(b) $1440600 = 2^3 3^1 5^2 7^4$,

(c) n, where n is squarefree,

(d) $p_1^2 p_2^2 \dots p_n^2$, where $p_1, \dots, p_n$ are pairwise distinct prime numbers.

Question 3.

Denote by $\pi(n)$ the set of prime divisors of some natural number n. Let G be a finite abelian group with primary decomposition

$$G \cong \bigoplus_{p \in \pi(|G|)} \left[\bigoplus_{i=1}^{\beta_p} \mathbb{Z}_{p^{e_i}} \right].$$

Describe $\mu = \max\{|g| : g \in G\}$ and prove your statement. Then, show that $|a|$ divides μ for every $a \in G$.

Quotients and isomorphism theorems.

As stated during the last review session, you may use without proof the lemmas that a subgroup is normal if and only if it is the kernel of some homomorphism and that $aH = bH \Leftrightarrow b^{-1}a \in H$. You may also cite the first isomorphism theorem, though you should be familiar with the proofs of all three (especially the first two).

Question 4.

Suppose that $H, K \trianglelefteq G$, $|G| = |H||K|$, and $H \cap K = 1$. Prove that $G \cong H \oplus K$ and $G/K \cong H$.

Question 5.

For each of the following groups G, explicitly compute G/K for every subgroup $K \leqslant G$, including the primary decomposition.

(a) $\mathbb{Z}_2$.

(b) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(c) $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Conclude that the following statement is false, and provide a counterexample:

"If K_1, K_2 are isomorphic subgroups of a group G, then G/K_1 is isomorphic to G/K_2."

Question 6.

Show that $\mathrm{SL}_n(\mathbb{R}) \trianglelefteq \mathrm{GL}_n(\mathbb{R})$. Describe $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ and prove your statement.