1) Heat Conduction Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 T}{\partial \phi^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3}$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial V}{\partial \mu}\right] + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial V}{\partial t} \tag{4}$$

2) Solutions and transformations for spherical coordinates

$\mu = \cos(\theta)$

T = T(r, t) $\Rightarrow$ Let U(r, t) = r T(r, t) and Equation (3) becomes $\frac{\partial^2 U}{\partial r^2} + \frac{rg}{k} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$.

T = T(r, μ, t) $\Rightarrow$ Let V = $r^{1/2}$ T yields Equation (4)

T = T(r, μ, ϕ, t) $\Rightarrow$ Let V = $r^{1/2}$ T yields Equation (4)

For U = rT: $T(r \to 0) = \textit{finite}$ <u>becomes</u> U(r=0) = 0 & $U'(r \to 0) = \textit{finite}$

$\frac{\partial T}{\partial r}$ <u>becomes</u> $\frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{r^2}U$

For V = $r^{1/2}$ T: $T(r \to 0) = \textit{finite}$ <u>becomes</u> $V(r \to 0) = \textit{finite}$

$\frac{\partial T}{\partial r}$ <u>becomes</u> $\frac{1}{r^{1/2}}\frac{\partial V}{\partial r} - \frac{1}{2}\frac{V}{r^{3/2}}$

3) ODEs

$r^2 R'' + raR' + bR = 0$ yields the auxiliary equation $\lambda^2 + (a-1)\lambda + b = 0$

Two real roots: $R(r) = C_1 r^{\lambda_1} + C_2 r^{\lambda_2}$

Double root: $R(r) = C_1 r^{\lambda} + C_1 \ln(r) r^{\lambda}$

$X'' + \lambda^2 X = 0$ yields $X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$

$X'' - \lambda^2 X = 0$ yields $X(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x)$

or $X(x) = C_1 \exp(\lambda x) + C_2 \exp(-\lambda x)$

$R'' + \frac{1}{r} R' + (\lambda^2 - \frac{\nu^2}{r^2}) R = 0$ yields $R(r) = C_1 J_\nu(\lambda r) + C_2 Y_\nu(\lambda r)$

$R'' + \frac{1}{r} R' - (\lambda^2 + \frac{\nu^2}{r^2}) R = 0$ yields $R(r) = C_1 I_\nu(\lambda r) + C_2 K_\nu(\lambda r)$

$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right] + n(n+1)M = 0$ yields $M(\mu) = C_1 P_n(\mu) + C_2 Q_n(\mu)$

for integer n = 0, 1, 2, 3 …..

$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]M = 0$ yields $M(\mu) = C_1 P_n^m(\mu) + C_2 Q_n^m(\mu)$

for integer n = 0, 1, 2, 3 ….. and integer m, with m less than n.

$\frac{r^2}{R}\left[R'' + \frac{1}{r} R' - \frac{1}{4}\frac{R}{r^2}\right] + \lambda^2 r^2 = n(n+1)$ yields $R(r) = C_1 J_{n+1/2}(\lambda r) + C_2 Y_{n+1/2}(\lambda r)$

4) Special Functions

Bessel Functions:

- $J_\nu(\lambda r)$ and $Y_\nu(\lambda r)$ are orthogonal with weighting function r.
- $J_o(0) = 1$ and $J_\nu(0) = 0$ for $\nu > 0$.
- $Y_\nu(0)$ goes to negative infinity for all ν.
- $I_\nu(\lambda r)$ and $K_\nu(\lambda r)$ are not orthogonal.
- $I_o(0) = 1$ and $I_\nu(0) = 0$ for $\nu > 0$.
- $K_\nu(0)$ goes to positive infinity for all ν.

Legendre Polynomials:

- $P_n^m(\mu) \quad and \quad P_n(\mu)$ are orthogonal with weighting function 1, for n and m both integers (0, 1, 2…).
- Note that $P_n^m(\mu) \to P_n(\mu)$ as m equals zero.
- Note that $P_n^m(\mu) = 0$ for m > n.
- $\int_{\mu=-1}^{1} P_n^m(\mu) P_k^m(\mu)\, d\mu = 0$ for all $n \neq k$. (valid for m = 0)
- $\int_{\mu=0}^{1} P_n(\mu) P_k(\mu)\, d\mu = 0$ for all $n \neq k$, provided both n & k are odd

  or both n & k are even.
- $P_o(\mu) = 1$, $P_2(\mu) = ½ (3\mu^2 – 1)$
- $P_1(\mu) = \mu$, $P_3(\mu) = ½ (5\mu^3 – 3\mu)$

Trigonometric Functions:

- $\cos(\nu x)$ and $\sin(\nu x)$ are orthogonal with weighting function 1 for the general characteristic value problem.
- Let $\nu = n$, with n = 0, 1, 2 … for 2π-periodicity.

5) Thermodynamics

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

$$\dot{Q} - \dot{W}_s = \frac{d}{dt}\int_{CV} e\rho dV + \int_{CS} (h + \frac{V^2}{2} + gz)\rho\vec{V} \cdot \hat{n} dA$$

$$\rho\frac{Dh}{Dt} = \frac{D\rho}{Dt} + \nabla \cdot (k\nabla T) + \tau_{ij}^{'}\frac{\partial u_i}{\partial x_j}$$

$$\tau_{ij}^{'} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla \cdot \vec{V}$$

$$Tds = dh - vdp$$

$$Tds = du + pdv$$

$$\eta_c = 1 - \frac{T_L}{T_H}$$

$$COP_c = \left(\frac{T_H}{T_L} - 1\right)^{-1}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = \left(\frac{v_1}{v_2}\right)^{k-1}$$

$$S = k \ln \Omega$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V$$

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_U$$

$$G = H - TS$$

$$A = U - TS$$

$$\frac{dp}{dT} = \frac{\Delta H}{T\Delta V}$$

Thermodynamics Continued

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$
$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$
$$\left(\frac{\partial T}{\partial V}\right)_p = -\left(\frac{\partial p}{\partial S}\right)_T$$
$$\left(\frac{\partial T}{\partial p}\right)_V = \left(\frac{\partial V}{\partial S}\right)_T$$

$$\Omega = \frac{(N+U-1)!}{(N-1)!U!}$$

6) Constitutive Equations

Ideal gas law: $PV = N\overline{R}T$

Volumetric thermal expansion coefficient: $\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$ where ρ is gas density.

Fourier's Law: $q_x'' = -k\frac{\partial T}{\partial x}$ (W/m^2)

Newton's Law of Cooling: $q_x'' = h(T - T_\infty)$ (W/m^2)

Stream Functions and Blasius Solution:

$u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ are the velocity components.

$\Psi(x, y) = f(\eta)\sqrt{\nu U x}$ where U is the free-stream velocity and ν is the kinematic viscosity,

with $\eta = y\sqrt{U/\nu x}$

7) Incompressible Fluid

Conservation of Mass

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Conservation of Momentum

x-momentum

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+\mathrm{X}$$

y-momentum

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial v}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]+Y$$

Conservation of Energy

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]+\mu\Phi+q'''$$

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left[\left(\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)\right]^2$$

$$\mathrm{Re}=\frac{ux}{\nu}$$

$$\mathrm{Pr}=\frac{\nu}{\alpha}$$

$$Nu=\frac{hx}{k}$$

$$St=\frac{h}{\rho c_p u}$$