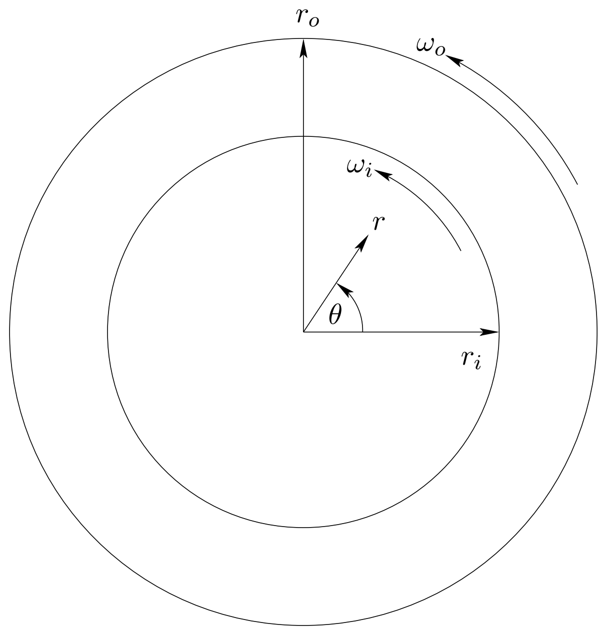
EGM 6813 - Fluid Mechanics 2
Fall 2007
Qualifying Exam

1. Consider the steady incompressible laminar flow between two concentric cylinders of radii r_i and r_o generated by the constant rotation of one or both cylinders at angular velocities ω_i and ω_o, as illustrated below.



   (a) *(4 points)* State clearly any appropriate assumptions and simplify the steady incompressible Navier-Stokes equations in polar coordinates (given below) accordingly.

   (b) *(7 points)* Assuming a solution of the form $u_\theta = Cr^\alpha$ show that the simplified θ-momentum equation is satisfied by $\alpha_1 = 1$ and $\alpha_2 = -1$. Using this result and the appropriate boundary conditions show that the solution of the simplified θ-momentum equation can be expressed as

$$u_\theta = \omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o} + \omega_o r_o \frac{r/r_i - r_i/r}{r_o/r_i - r_i/r_o}. \tag{1}$$

   (c) *(1 point)* Explain why the viscosity does not appear in Eq. (1).

   (d) *(2 points)* In Eq. (1), take the limit of $r_i \to 0$ and $\omega_i \to 0$ and comment on the resulting expression.

   (e) *(2 points)* In Eq. (1), take the limit of $r_o \to \infty$ and $\omega_o \to 0$ and show that comment on the resulting expression.

   (f) *(4 points)* Show that the torque per unit width on the inner cylinder is

$$T_i = 4\pi\mu \frac{r_i^2 r_o^2}{r_o^2 - r_i^2}(\omega_o - \omega_i), \tag{2}$$

and that the torque per unit width on the outer cylinder is

$$T_o = -T_i. \tag{3}$$

How do you envisage these expressions could be useful in practice?

The steady incompressible Navier-Stokes equations in polar coordinates are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{\theta r}}{\partial \theta} - \tau_{\theta\theta}\right]$$

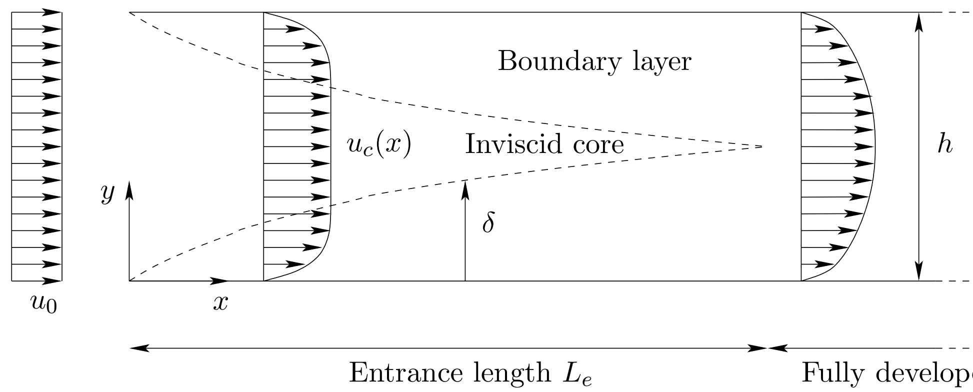
$$u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \tau_{\theta r} - \tau_{r\theta}\right]$$

$$\tau_{rr} = 2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]$$

2. Consider the planar incompressible laminar flow in the entrance region between two infinitely thin flat plates a distance h apart, as depicted schematically in the figure below. In the entrance region, i.e., $0 \leq x \leq L_e$ where L_e is the so-called entrance length, the boundary layers growing on the two plates are separated by the inviscid core. The flow becomes fully developed when the boundary layers merge at the center line. Upstream of the plates, the velocity is u_0.



Assume that the velocity in the inviscid core is $u_c(x)$ and that the velocity in the boundary layer can be approximated by the parabolic profile

$$\frac{u(x,y)}{u_c(x)} = \frac{y}{\delta}\left(2 - \frac{y}{\delta}\right), \tag{4}$$

where $\delta(x)$ denotes the boundary-layer thickness.

(a) *(3 points)* Use conservation of mass to show that the boundary-layer thickness increases according to

$$\frac{\delta}{h} = \frac{3}{2}\left(1 - \frac{u_0}{u_c}\right). \tag{5}$$

(b) *(3 points)* Show that, for the profile given by Eq. (4), the displacement thickness δ^*, momentum thickness θ, and the skin-friction coefficient C_f are given by

$$\frac{\delta^*}{\delta} = \frac{1}{3} \quad \text{and} \quad \frac{\theta}{\delta} = \frac{2}{15} \quad \text{and} \quad C_f = \frac{4\nu}{u_c \delta}.$$

(c) *(8 points)* Use the results you derived in (a) and (b) and the momentum-integral equation (which you do *not* need to derive)

$$\frac{C_f}{2} = \frac{d\theta}{dx} + (H+2)\,\frac{\theta}{u_c}\frac{du_c}{dx}, \tag{6}$$

to show that the non-dimensional velocity in the inviscid core, defined as $\overline{u}_c = u_c/u_0$, satisfies

$$\left(9 - \frac{16}{\overline{u}_c} + \frac{7}{\overline{u}_c^2}\right)\frac{d\overline{u}_c}{d\overline{x}} = \frac{40}{3}\frac{\nu}{u_0 h}, \tag{7}$$

where $\overline{x} = x/h$.

(d) *(6 points)* Show that the entrance length is given by

$$\frac{L_e}{h} = 0.0259\frac{u_0 h}{\nu}. \tag{8}$$

Discuss, from a physical point of view, whether the dependence of L_e on u_0, h, and ν makes sense. (That is, should L_e/h increase with u_0 or not, and similarly for the other quantities.)