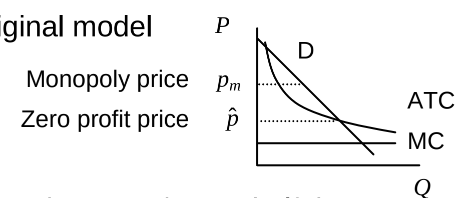
# Homogeneous Product Oligopoly

**Industrial Organization** - study of markets without perfect competition
  **Results** aren't very general because they're sensitive to model assumptions (which results from real world being complicated, not from "bad" modeling)
  **Issues** - (1) Choice between market and within-firm activity (level of vertical integration); covered in Jamison's course
    (2) Antitrust and regulation; covered in agency theory (Sappington's course)
    (3) Production technology's impact on market performance; used to be called "structure-conduct-performance" paradigm; mostly empirical work (covered in Ai's course)
    (4) R&D/Innovation as source of market power; worry about whether it's temporary or permanent?
**Firm** - "black box" represented by cost function and strategy set
**PSNE** - pure strategy Nash equilibrium
**MSNE** - mixed strategy Nash equilibrium

## Bertrand Model - simultaneous price setting by oligopolists who then produce as needed to satisfy demand generated by that price

  **Assumptions** - homogeneous good; constant MC and no fixed cost (implies constant ATC)
  **Result** - PSNE: $p$ = MC = ATC (number of firms doesn't matter; ≥ 2)
  **Bertrand Paradox** - huge monopoly/duopoly discontinuity; go from monopoly output to perfect competition ($p$ = MC) by going from 1 firm to 2 firms; Hamilton said people who call it a paradox "don't understand the Bertrand model"
**How General** - loosen/change assumptions and results change
  **Increasing Returns** - decreasing MC; can't have $p$ = MC and $\pi \geq 0$ because ATC > MC (i.e., ATC is above MC when MC is decreasing); firms will be losing money at the standard Bertrand equilibrium so it's not an equilibrium here (they always have the choice of zero profit)
  **Fixed Cost** - add small fixed cost, but keep constant MC from original model



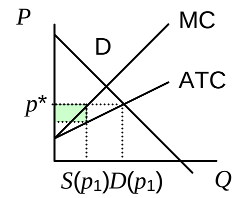

$$C(q) = \begin{cases} cq + k & q > 0 \\ 0 & q = 0 \end{cases}$$ ($k$ = fixed cost; $c$ = MC)

    **Best Reply** - look at firm 1's best reply to firm 2's price
      $p_2 \leq \hat{p}$ - firm 1 shouldn't produce; below $\hat{p}$ there will be negative profits; at $\hat{p}$, firm 1 will have to split the market (e.g., each gets half the customers) so it'll be below ATC (i.e., negative profit)
      $\hat{p} < p_2 \leq p_m$ - set $p_1 = p_2 - \varepsilon$ (i.e., slightly less than firm 2's price); firm 1 will capture entire market and be above ATC
      **Result** - don't get PSNE; original equilibrium ($p$ = MC) won't happen because MC is always below ATC in this scenario
  **Increasing MC** - Tirole "cheats" by implicitly assuming firms are free to choose not to meet demand if the cost is too high (i.e., low price firm rations consumers and other firm faces residual demand)

    **ATC** - assume MC is linear (i.e., MC $= k_1 + k_2 q$); $C(x) = \int_0^x (k_1 + k_2 q)dq = k_1 x + \frac{k_2}{2}x^2 + k_3$

      $\therefore$ ATC $= \frac{C(x)}{x} = k_1 + \frac{k_2}{2}x + k_3$ ... same intercept and half the slope as MC

**$p_1 < p_2$** (assumption) - $D(p_1)$ customers show up to buy from firm 1, but firm 1 only wants to sell $S(p_1)$ so firm "rations" consumers (chooses who to serve)




**Divide Equally** - take $p = p^*$ as given; $S_1(p^*) + S_2(p^*) = D(p^*)$; both firms have $\pi_i > 0$, but $p_1 = p^*$ is not a best reply for $p_2 = p^*$ (i.e., this is not a PSNE)

Proof: assume $p_2 = p^*$ and firms are allowed to ration $\therefore$ firm 2 sells $S(p^*)$

Since $D(p^*) > S(p^*)$, there is excess demand

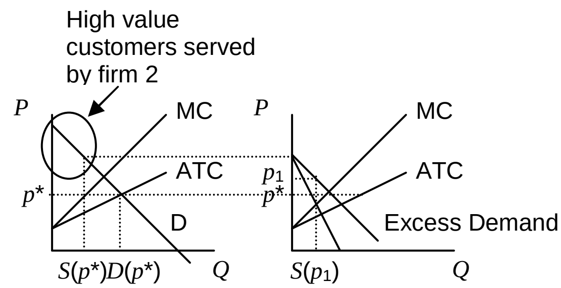
$\therefore$ $\exists$ some customers willing to pay more than $p^*$... "non-Slutsky assumption" if firm 1 charges a price greater than $p^*$, all customers don't disappear (i.e., there is still demand at the higher price)

Assume **efficient/parallel rationing** - firm 2 serves the high value customers

From graphs it's obvious firm 1 is better off charging $p_1 > p^*$

Add to that, if firm 1 charges more ($p_1$), then firm 2 doesn't want to charge $p^*$

$\therefore$ $p_1 = p_2 = p^*$ is not a PSNE



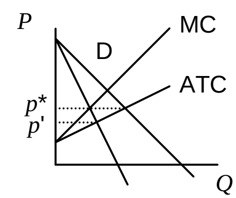

**Cournot Model** - choose quantity and "auctioneer" sells all output at market price; firm has no opportunity to back out; $\pi_i = P(q_1 + q_2)q_i - C(q_i)$

**Modified Bertrand Game** - firms post a price and must then sell to all customers who want to buy; results are similar to Cournot model

$p_1 < p_2 \;\Rightarrow\; D_1 = D(p_1)$ (firm 1 gets all the demand)

**Mathanese** - $p_1 = p_2 \;\Rightarrow\; D_1 = \frac{1}{2}D(p_1)$ (firms evenly split the demand)

$p_1 > p_2 \;\Rightarrow\; D_1 = 0$ (firm 1 doesn't get any demand)




**Scenarios** - (1) $p_1 > p_2$... firm 1 sells nothing $\pi_1 = 0$

(2) $p_1 < p_2$... firm 1 faces all demand; profit depends:

$p_1 > p^* \;\Rightarrow\; \pi_1 > 0$

$p_1 = p^* \;\Rightarrow\; \pi_1 = 0$

$p_1 < p^* \;\Rightarrow\; \pi_1 < 0$

(3) $p_1 = p_2 = p^*$... firms split market with MC = MR > ATC; equilibrium with $\pi_1 = \pi_2 > 0$

(4) $p_1 = p_2 = p'$... firms split market with MC > MR = ATC; equilibrium with $\pi_1 = \pi_2 = 0$

**Result** - any $p_1 = p_2$ between $p'$ and $p^*$ is an equilibrium

Proof: assume we start at $p_1 = p_2 = p''$ with $p' < p'' < p^*$, would firm 1 ever change its price?

$p_1 > p''$... no because firm 1 won't sell anything

$p_1 < p''$... no because firm 1 would get all demand and be well below ATC (i.e. negative profit)

**Ration?** - would firms choose to commit to not ration?

Not Commit... get MSNE, "unique, but may not be calculable"

Commit... get continuum of PSNE; can't just pick one ("hand waving")

**Result** - can't directly compare; we may revisit this in product differentiation

## **Existence of Nash Equilibrium** - should prove equilibrium exists before looking at results of that equilibrium

**Finite Strategy Space** - Nash's Theorem says we always have equilibrium (at least in mixed strategy; may not have a PSNE)

**Continuous Strategy Space** - examples include prices, quantities, locations; we're only interested in PSNE

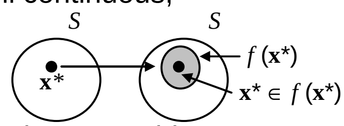
**Classic Approach** - find best reply functions with "nice" properties and apply a fixed point theorem (Brouwer or Kakutani)

(from ECO 7120 notes)

**Brouwer Fixed Point Theorem** - if $f: S \to S$ (i.e., $f$ is a mapping from set $S$ into itself) is continuous and set $S$ is compact (closed & bounded) and convex, then $\exists\ \mathbf{s}^* \in S$ with $f(\mathbf{s}^*) = \mathbf{s}^*$ (i.e. a point that maps to itself)

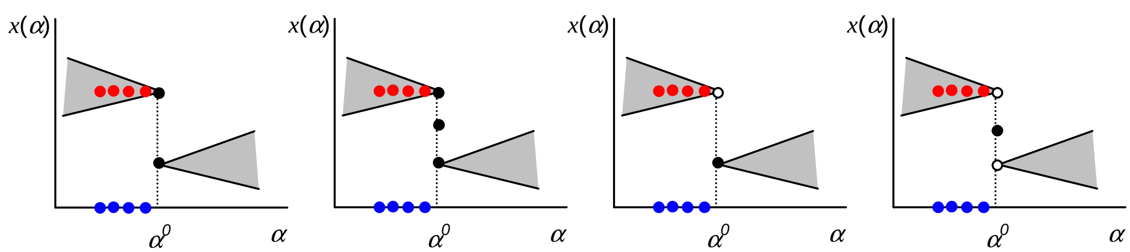
**Kakutani Fixed Point Theorem** - if $f: S \to 2^S$ (i.e., $f$ is a mapping from set $S$ into the set of all subsets of itself) is compact valued, convex valued, and upper hemi continuous, and $S$ is compact and convex, then $\exists\ \mathbf{x}^* \in S$ with $\mathbf{x}^* \in F(\mathbf{x}^*)$

(This is a generalization of Brouwer FPT)




**Upper Hemi Continuity** - this is the "sort of" continuous we talked about in micro; Consider sequence of points $\alpha^n$ that converges to $\alpha^0$ (blue dots in graphs); upper hemi continuity says that any series determined by $x(\alpha^n)$ (red dots) converges to a point in $x(\alpha^0)$

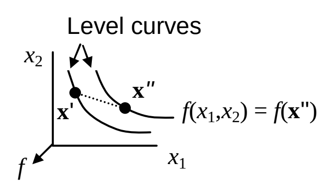
Are UH Continuous | Are Not UH Continuous




**Formally** - given the convergent sequence $\alpha^n \to \alpha^0$, then any sequence $y^n \in x(\alpha^n)$, with $y^n \to \bar{y}$ has $\bar{y} \in x(\alpha^0)$

**Another Way** - if sequence of points in the correspondence converges to $(\alpha^0, \bar{y})$, then $(\alpha^0, \bar{y})$ must be in the correspondence

(from ECO 7115 Notes)



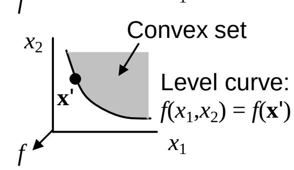

**Quasiconcave** - two definitions

1) Chord connecting two points in domain lies above level curve of one of the original points

$\forall\ \mathbf{x}'$ and $\mathbf{x}'' \in D$ and $\lambda \in (0,1)$, $f(\lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'') \geq \min[f(\mathbf{x}'), f(\mathbf{x}'')]$

2) Set of all points greater than a level curve is convex

$\{\mathbf{x} : f(\mathbf{x}) \geq f(\mathbf{x}')\}\ \forall\ \mathbf{x}' \in D$ is a convex set




**Importance** - guarantees maximizing value is single point or convex set of points (i.e., demand curve may have flat sections, but no jumps)

**Two Player Game** - example of showing existence

**Notation** -

$\mathbf{s}_i$ is a strategy for player $i$

$S_i$ is the set of possible strategies for player $i$ $\therefore$ $\mathbf{s}_i \in S_i$

$S \equiv S_1 \times S_2$ is the set of all possible strategy combinations for the two players (this is the set referred to in the fixed point theorems)

$\mathbf{s} \equiv (\mathbf{s}_1, \mathbf{s}_2) \in S$ is a specific set of strategies played

$\pi_i(\mathbf{s})$ is the payoff to player $i$ when both players play strategies in $\mathbf{s}$

**Needed** -

$S$ must be compact (closed & bounded) and convex

**Reality** - strategy space usually defined by prices for Bertrand games; prices have definite lower bound (zero), upper bound is usually constrained by demand... simple enough to impose an upper bound and then verify that it doesn't affect the solution

**Brouwer** - $\pi_i(\mathbf{s})$ is <u>concave</u> in player $i$'s strategy and <u>continuous</u> in every other player's strategies $\Rightarrow$ player $i$'s best reply is continuous so Brouwer FPT holds (at least one PSNE)

**Kakutani** - $\pi_i(\mathbf{s})$ is <u>quasiconcave</u> in player $i$'s strategy and <u>continuous</u> in every other player's strategies $\Rightarrow$ player $i$'s best reply is upper hemi continuous and convex valued so Kakutani FPT holds (at least one PSNE)

**Showing Global Concavity** - from ECO 7115 notes: Hessian is negative definite (i.e., all n leading principal minors alternate sign: $|\mathbf{A}_1| < 0$, $|\mathbf{A}_2| > 0$, $|\mathbf{A}_3| < 0$, etc. (i.e., $(-1)^k |\mathbf{A}_k| > 0$); <u>OR</u> check all eigenvalues < 0

**Showing Global Quasiconcavity** - from ECO 7115 notes: determinant of the bordered Hessian is positive: $|\mathbf{BH}| > 0$; <u>OR</u> concavity implies quasiconcavity

**Vives Alternative** - discontinuities in best replies are not a problem if the jumps go in a particular direction

**Notation** -

$x_i \in X_i = [0, \bar{x}_i]$ (each firm has choice between zero & some max... output or price)

$X \equiv X_1 \times X_2$ (strategy space with 2 players)

$r_i(x_j)$ best reply correspondence for player $i$ when player $j$ chooses strategy $x_j$;

Note: firm $i$'s best reply must be defined $\forall\ x_j \in X_j$

**Best Reply Mapping** - $r(\mathbf{x}) = (r_1(x_2), r_2(x_1))$

**Fixed Point** - $r(\mathbf{x}^*) = \mathbf{x}^*$... defines a PSNE; $\mathbf{x}^*$ defines a pair of strategies that are best replies to themselves (definition of equilibrium)

**Using Graphs** - assume best replies are close so at any jump point there are 2 (or more) values in the best reply

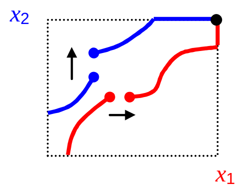
**Horizontal or Vertical?** - given requirement for best reply stated above ($r_i(x_j)$ is defined $\forall\ x_j \in X_j$), that means player 1's best reply has a value for every value of $x_2$ so 1's best reply spans entire vertical range in graph; also, jumps in player 1's best reply must be horizontal (no vertical gaps); similarly, player 2's best reply has a value for every value of $x_1$ so 2's best reply spans entire horizontal range in graph and jumps must be vertical (no horizontal gaps)

**No Jumps** - it's clear that the best reply functions must intersect at least once

**Downward Jumps** - may not intersect

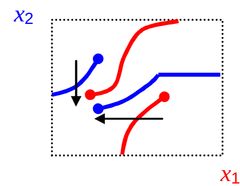
**Upward Jumps** - must still intersect

**Upward Jumps**




PSNE. In this case, it's at upper bound so may worry if the upper bound was imposed

**Downward Jumps**




No PSNE.

**Monotone Increasing** - Vives established that for <u>two or more</u> players each with <u>multiple</u> strategic variables, if the best replies are <u>upward sloping</u> everywhere they are continuous and all discontinuities (jumps) are <u>upward</u>, then there must exist a fixed point (i.e., PSNE)

**Monotone Decreasing** - Vives established that if there are <u>only two</u> players each with a <u>single</u> strategic variable, if the best replies are <u>downward sloping</u> everywhere they are continuous and all discontinuities (jumps) are <u>downward</u>, then there must exist a fixed point (i.e., PSNE)

**Showing Monotone Inc/Dec** - need to find slope of best reply (i.e., $dx_1/dx_2$)

**Best Reply** - solves $\max_{x_1 \in X_1} \pi_1(x_1, x_2)$

Solve... $\dfrac{\partial \pi_1}{\partial x_1} = 0$

<u>Lucky</u>... invert and solve to get $x_1{}^* = f_1(x_2)$

<u>Unlucky</u>... totally differentiate: $\dfrac{\partial^2 \pi_1}{\partial x_1^2} dx_1 + \dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2} dx_2 = 0$

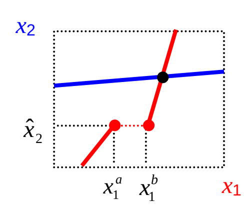
Then solve $\dfrac{dx_1}{dx_2} = -\dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \bigg/ \dfrac{\partial^2 \pi_1}{\partial x_1^2}$

$\pi_1$ concave $\Rightarrow \dfrac{\partial^2 \pi_1}{\partial x_1^2} < 0 \;\therefore$ sign of $\dfrac{dx_1}{dx_2}$ is same as sign of $\dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_1}$

(this works for globally concave or locally concave near local max)

$\therefore$ upward sloping means $\dfrac{\partial^2 \pi_1}{\partial x_1 \partial x_1} > 0$

**PS 1.5** - look at upward sloping best replies and upward jumps




Consider both parts of player 1's best reply as separate functions:

$x_1^a(x_2)$ and $x_1^b(x_2)$

At $\hat{x}_2$ profit for player 1 is the same so we have:

$F = \pi_1(x_1^b(\hat{x}_2), \hat{x}_2) - \pi_1(x_1^a(\hat{x}_2), \hat{x}_2) = 0$

Differentiate wrt $x_2$ (still evaluating at $\hat{x}_2$):

$$\left.\frac{\partial F}{\partial x_2}\right|^{x_2=\hat{x}_2} = \frac{\partial \pi_1}{\partial x_1}\frac{\partial x_1^b}{\partial x_2} + \frac{\partial \pi_1(x_1^b(\hat{x}_2),\hat{x}_2)}{\partial x_2} - \frac{\partial \pi_1}{\partial x_1}\frac{\partial x_1^a}{\partial x_2} - \frac{\partial \pi_1(x_1^a(\hat{x}_2),\hat{x}_2)}{\partial x_2}$$

Since $\pi_1$ is a local max at $\hat{x}_2$, we know $\partial\pi_1/\partial x_1 = 0$ (drops terms 1 and 3)

$$\left.\frac{\partial F}{\partial x_2}\right|^{x_2=\hat{x}_2} = \frac{\partial \pi_1(x_1^b(\hat{x}_2),\hat{x}_2)}{\partial x_2} - \frac{\partial \pi_1(x_1^a(\hat{x}_2),\hat{x}_2)}{\partial x_2}$$

Given the upward jump, this difference must be positive. Below $\hat{x}_2$, the best reply is $x_1^a(x_2)$ so we know $\pi_1(x_1^a(x_2),x_2) > \pi_1(x_1^b(x_2),x_2)$ (i.e., $F<0$). Above $\hat{x}_2$, the opposite is true: the best reply is $x_1^b(x_2)$ so we know $F>0$. So in the neighborhood around $\hat{x}_2$, as we increase $x_2$, the function $F$ goes from being negative to positive. That means it is increasing so

$$\left.\frac{\partial F}{\partial x_2}\right|^{x_2=\hat{x}_2} = \frac{\partial \pi_1(x_1^b(\hat{x}_2),\hat{x}_2)}{\partial x_2} - \frac{\partial \pi_1(x_1^a(\hat{x}_2),\hat{x}_2)}{\partial x_2} > 0$$

By the fundamental theorem of calculus we can rewrite this as an integral:

$$\left.\frac{\partial F}{\partial x_2}\right|^{x_2=\hat{x}_2} = \int_{x_1^a}^{x_1^b} \frac{\partial^2 \pi_1(x_1(\hat{x}_2),\hat{x}_2)}{\partial x_1 \partial x_2} dx_1 > 0$$

Since we know $x_1^b > x_1^a$, the above statement implies $\frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} > 0$

**Summary** -

Started with concavity of payoffs

Relaxed to quasi-concavity

Relaxed to upward sloping with upward jumps (monotone increasing)

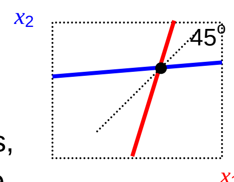
We can check this rather than global concavity; may be easier to check and may hold when concavity (or quasi-concavity) fail

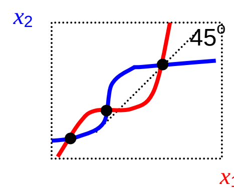
## Uniqueness of Nash Equilibrium -

**Easy Rule** - if player 1's best reply always crosses steeper than player 2's and the best replies are continuous, then there's a unique Nash equilibrium

**Crosses** - means a Nash equilibrium exists because best replies intersect

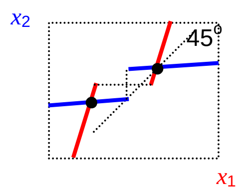
**Steeper** - $\frac{\partial R_1}{\partial x_2} < 1$ and $\frac{\partial R_2}{\partial x_1} < 1$




This is harder to see; look at the 45° line; the best reply for 2 (blue) increases as $x_1$ increases, but stays below the 45° line $\therefore$ $\partial R_2/\partial x_1 < 1$; we can make the same argument for 1's best reply, but realize that we have to look at the transpose of the graph

**Continuous** - without continuity this rule doesn't hold



**Using Concavity** -

**Best Replies** - $x_1^*(x_2) \equiv \arg\max_{x_1} \pi_1(x_1, x_2)$

**Increasing Monotonic Transformation** - of $\pi_1$ doesn't change $x_1^*(x_2)$ $\therefore$ if we don't' have concavity, try a transformation of $\pi_1$

**$\ln$ Transformation** - $\ln$ is good to use because it's monotone increasing and it allows us to break up products

**Example** - $\ln(\pi_1) = \ln[(p_1 - c)D_1(p_1, p_2)] = \ln((p_1 - c) + \ln D_1(p_1, p_2)$

For concavity, need to show $\frac{\partial^2 \ln(\pi_1)}{\partial p_1^2} < 0$

$$\frac{\partial \ln(\pi_1)}{\partial p_1} = \frac{1}{(p_1 - c)} + \frac{1}{D_1(p_1, p_2)} \frac{\partial D_1}{\partial p_1}$$

$$\frac{\partial^2 \ln(\pi_1)}{\partial p_1^2} = \frac{-1}{(p_1 - c)^2} + \frac{1}{D_1(p_1, p_2)} \frac{\partial^2 D_1}{\partial p_1^2} + \frac{-1}{[D_1(p_1, p_2)]^2} \left(\frac{\partial D_1}{\partial p_1}\right)^2$$

The first term is negative so a sufficient condition would be for $\ln D_1$ to be concave

Checking $\ln D_1$:

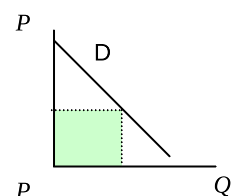
1) $\frac{1}{D_1(p_1, p_2)} \frac{\partial^2 D_1}{\partial p_1^2} + \frac{-1}{[D_1(p_1, p_2)]^2} \left(\frac{\partial D_1}{\partial p_1}\right)^2$, or

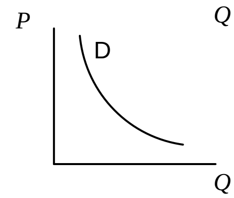
2) look at $\ln D_1$

**Trick** - redefine strategic variable as price - cost

**Linear Demand** - know there's a max profit (revenue) at midpoint; maximizes area of rectangle under the line

**Concave Demand** - taking $\ln$ will make it "less concave"




## Cournot with Asymmetric Firms

Tirole does linear demand and different, constant marginal cost; we'll drop the linear demand assumption

**Inverse Demand** - $P(Q)$, where $Q = \sum_{i=1}^{n} q_i$ (don't forget to use this for derivatives)

**Profit** - $\pi_i(\tilde{\mathbf{q}}) = P(Q) \cdot q_i - c_i q_i$; market price as function of total output times firm's output (i.e., total revenue) minus unit cost times firm's output (i.e., total cost)

Can also write this as $\pi_i(q_i, Q)$ or $\pi_i\left(q_1, \sum_{j \neq i} q_j\right)$;

$\pi_i(q_i, Q)$ is better, "but we're not going to worry about that today"

**First Order Conditions** - $\frac{\partial \pi_i}{\partial q_i} = P(Q) + q_i \frac{dP}{dQ} - c_i = 0 \;\; \forall \;\; i = 1...n$ (assuming all firms produce)

Rewrite: $P(Q) + q_i \frac{dP}{dQ} = c_i$ for the math challenged: $\frac{\partial P(Q)}{\partial q_i} = \frac{dP(Q)}{dQ} \cancelto{1}{\frac{\partial Q}{\partial q_i}} = \frac{dP(Q)}{dQ}$

Sum for all firms: $nP(Q)+Q\frac{dP}{dQ}=\sum_{i=1}^{n}c_i$

**Trick** - without this summation trick, we'd have $n$ simultaneous equations

Divide by $n$: $P(Q)+\frac{Q}{n}\frac{dP}{dQ}=\frac{1}{n}\sum_{i=1}^{n}c_i$

RHS - average unit cost
LHS - similar to marginal revenue

If $n=1$ we have $P(Q)+Q\frac{dP}{dQ}=\frac{\partial P(Q)Q}{\partial Q}=\frac{\partial \mathrm{TR}}{\partial Q}=\mathrm{MR}$

**Result** - LHS is function of aggregate demand and aggregate market quantity only (and number of firms, but we'll assume that doesn't change) $\therefore$ if we change costs while holding the average constant, there is no change in $P^*(Q^*)$ or $Q^*$

**Future Problem Set** - shift everyone's cost the same way (e.g., tax)

## Mixed Strategy Equilibria

**Why** - failure of quasiconcavity; payoff discontinuities (with continuous strategy space... presented by Hotelling; we'll cover it later)

**Mixed Strategy** - recall from game theory that all expected profits must be equal for all pure strategies played with positive probability (and those not used must be less)

**Distribution Function** - in general, we'll set up a differential equation to solve for the distribution function of the strategy; endpoints come from upper and lower bounds for prices (from dominated strategy arguments)

**Varian** (AER 1980) -

Free entry
Strictly declining ATC
Consumers: (1) buy at most 1 unit; (2) reservation price $r$; (3) 2 types:

**Informed** - $I$ of them; know all prices so they buy at the low price store
**Uninformed** - $M$ of them; choose a store at random

$U=\frac{M}{n}=$ number of uninformed customers in each store (assumes $n$ stores)

**Sales** - lowest price firm: $I+U$

Not lowest price firm: $U$
Tie with $k$ firms for lowest price: $I/k+U$ (won't be any ties)

**Density** - $f(p)$ is density function under which firm chooses price

$p^*=\frac{C(I+U)}{I+U}=$ ATC of selling to max customers

**Properties of *f* (*p*)** -

(1) $f(p)=0$ for $p>r$ and $p<p^*$ (zero or negative profit in those regions)

(2) No symmetric equilibrium with all firms charging the same price

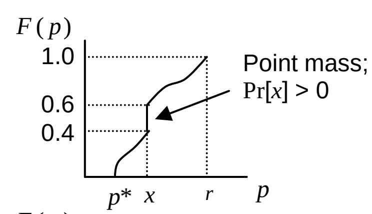
Proof: sales would be $I/n+U$

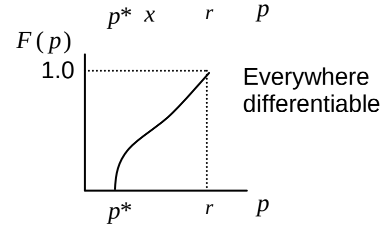
Firm can cut price by $\varepsilon$ to get sales up to $I+U$
Firm cut raise price a lot (up to $r$) and get sales to $U$

(3) No point masses in interior of distribution (between $r$ and $p^*$)

Proof: must have $p>p^*$

Point mass means positive probability of a tie

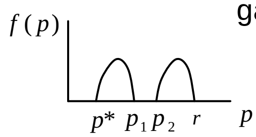


One firm could charge $p-\varepsilon$ and raise its profits

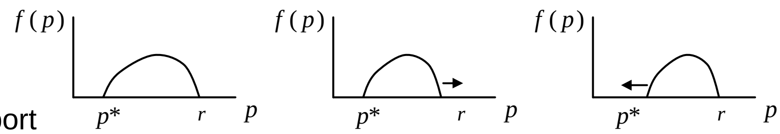
Can't tie at $p^*$ because firms would have negative profit

(4) No gaps where $f(p)=0$ in interior




gap will be filled in because anyone charging $p_1$ would rather charge $p_2$

(5) Support from $p^*$ to $r$




First graph shows full support

Second graph: if firm isn't lowest price, it would be better off charging $r$

Third graph: firms can always charge less (up to $p^*$) and get more sales

(6) All pure strategies in a mixed strategy have equal expected payoff

<u>Proof</u>: Free entry means expected payoff = 0

Only two states: lowest price and not

Probability of being lowest price ($S$): $[1-F(p)]^{n-1}$

Expected profit:

$$\underbrace{\pi_S(p)}_{+}\underbrace{[1-F(p)]^{n-1}}_{+}+\underbrace{\pi_F(p)}_{-}\underbrace{\left[1-(1-F(p))^{n-1}\right]}_{+}=0$$

(make profit when low price; negative profit if not low price)

Solve for $F(p)$: $F(p)=1-\left[\dfrac{\pi_F(p)}{\pi_F(p)-\pi_S(p)}\right]^{\frac{1}{n-1}}$

**Solve *f* (*p*)** - $f(p)=\dfrac{\partial F(p)}{\partial p}$

Usually have to do this with differential equation; then find end points and integrate up

**Problem Set** - will have problem similar: Cournot model with discontinuity (sunk cost)... enough to mess up quasiconcavity

**Supermodularity** - new approach to existence and comparative statics; Vives Chpt 1 covers fundamentals; Amir covers sample cases

**Lattices** - "extremely abstract mathematical structures"; topology for fixed point theorems easier to prove this way (but harder to understand)

**Strategic Complementarity** - type of problem this technique is good for

**Maximization Problem** - $\max_a F(a,s)$ s.t. $a \in A_s$

- **Parameter** - $s \in S$
- **Constraint Set** - $A_s$ (varies with parameter $s$; similar to consumer budget problem; changing parameter (prices) changes feasible region (budget set))
- **Variable** - $a \in A_s$
- **Maximized Value** - $a^*(s) = \arg\max_a \{F(a,s) : a \in A_s\}$
  - Note: $a^*(s)$ could have multiple values
- **Supermodular** - "strictly increasing differences"... we'll assume this property holds:
  - $F(a',s') - F(a,s') > F(a',s) - F(a,s) \quad \forall \quad s' > s$ and $a' > a$
  - This is equivalent to $\dfrac{\partial^2 F}{\partial s \partial a} > 0$ although we haven't said anything about differentiability)
- **Increasing Maxima Theorem** - if (a) $F$ has increasing differences in $(a,s)$, (b)
  - $A_s = [g(s), h(s)]$ (a closed interval with $g(s) \le h(s)$), and (c) $g(s)$ and $h(s)$ are increasing functions, then the minimum and maximum values of $a^*(s)$ are increasing functions
  - If $F$ has <u>strictly</u> increasing differences then every value of $a^*(s)$ is increasing
  - **Comparative Statics Result** - shows how a change in $s$ changes the maximized value; but using the old comparative statics technique we needed assumptions of concavity or quasiconcavity
    - **Example** - if $F$ if differentiable and we have interior solutions:
      - First Order Condition: $\dfrac{\partial F(a,s)}{\partial a} = 0$
      - Totally differentiate: $\dfrac{\partial^2 F(a,s)}{\partial a^2} da + \dfrac{\partial^2 F(a,s)}{\partial a \partial s} ds = 0 \quad \Rightarrow \quad \dfrac{da}{ds} = -\dfrac{\partial^2 F(a,s)/\partial a \partial s}{\partial^2 F(a,s)/\partial a^2}$
      - Concavity: if we assume $\dfrac{\partial^2 F(a,s)}{\partial a^2} < 0$, then the sign of $\dfrac{da}{ds}$ will be the same as the sign of $\dfrac{\partial^2 F(a,s)}{\partial a \partial s}$
  - **New Way** - with new technique we don't need the concavity assumption; increasing differences ensures $\dfrac{\partial^2 F(a,s)}{\partial a \partial s} > 0$ so we know $\dfrac{da}{ds} > 0$ (that's what the theorem above says)
  - **Boundary Solutions** - if $a^*(\hat{s}) = g(\hat{s})$ (i.e., boundary solution), the result still holds because $g(s)$ is increasing in $s$

**Supermodular Games** - assume $a =$ own action, $s =$ everyone else's actions, and all payoffs are supermodular

$s$ in previous notation (other people's actions are the parameters that determine the feasible set)

Set of players: $N = \{1, 2, \ldots, n\}$

Payoff function: $F_i(\mathbf{a}) \subset R$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$

Assume payoff functions have increasing differences in player's own action and each rival's action (i.e., $F$ has increasing differences in $(a, s)$)

Assume action sets are compact

Assume best replies are well defined (guaranteed if $F_i(\mathbf{a})$ is upper hemicontinuous)

**Results** - (1) $\exists$ at least one pure strategy Nash equilibrium

(2) if it's unique, it's **dominance solvable** - using successive elimination of strictly dominated strategies; stronger property than Nash equilibrium because players can find it (no coordination problem)

(3) largest and smallest PSNE are increasing functions of parameter

**Applying Supermodularity** - we'll see some of these later

Bertrand with differentiated products

**Cournot Trick** - $\pi_i(q_i, q_j)$ ... not supermodular because $q_j \uparrow$ eventually causes $\pi_i \downarrow$ (so does $q_i \uparrow$); relabeling the strategies can make this supermodular (but only works with 2 players)

Let $a_1 = q_1$ and $a_2 = -q_2$ (where $a_i$ is the "strategy" of player $i$)

Best Replies - decreasing with respect to old strategy ($q_i$), but increasing wrt new strategy ($a_i$)