Imperfect Information

Complete Information - all players know:
- Set of players
- Set of strategies for each player
- Outcomes as a function of the strategies
- Payoffs for each outcome (i.e., utility function for each player)

Incomplete Information - any of the four pieces of information above is unknown to a player; also called asymmetric information

Private Information - typical example of incomplete information where player knows something about his own actions that other players don't observe; could be the other way though (e.g., specialist [doctor, mechanic] has more information about your payoffs than you do)

Harsanyi - paper in Management Science (1967) on how to convert game with incomplete information to game of imperfect, but complete information by inserting random move by nature

Incomplete in Payoffs - Harsanyi first, argued that all types of incomplete information can be converted to incomplete information on payoffs

Players - not sure if player 1 is in the game... equivalent to knowing he is in the game, but not sure about his strategy space (e.g., could have four options or just one, effectively not in the game because he has no decisions)

Strategies - not knowing set of strategies could be transformed by adding strategies that will never be played because of \(-\infty\) payoff; now all options have same number of strategies, but we don't know which payoffs are correct

Outcomes - not knowing outcomes is equivalent to not knowing the utility functions (payoffs)... technically, there is a difference to knowing outcomes and not utility functions if there are several strategies (from different information sets) that have the same outcome (will yield the same payoffs)

Infinite Regress - in order to solve incomplete information problem, we have to worry about beliefs about beliefs about beliefs, etc.

Solution - Harsanyi solved problem by defining types of players, where players differ in payoffs (utility function) and/or beliefs

Simplification - Harsanyi also argued that all beliefs could be captured by a single probability distribution over the types of the other players (could be correlated by using a joint distribution)

Example - 3 types of "player 1" and 4 types of "player 2": \(t_1^1, t_2^1, t_3^1\) and \(t_1^2, t_2^2, t_3^2, t_4^2\)

Each type of "player 1" has beliefs about the probability that an opponent will be of a specific type:
- \(p_1^1 = (0.25, 0.25, 0.25, 0.25)\)
- \(p_1^2 = (0.1, 0.1, 0.1, 0.7)\)
- \(p_1^3 = (0.5, 0.5, 0, 0)\)
Fudge - in order to be tractable, this method requires a finite number of types

Standard - typically we only allow players types to differ on preference (utility function); theory allows preferences and/or beliefs to vary, but the full implications of this haven’t been studied

Problem? - may have assumed away the problem of incomplete information rather than solved it

Formally -

Players - \( n \) players

Strategies - each type of player \( i \) has the same strategy space \( S_i \) (total of \( n \) sets of types)

Set of Types - \( T_i \) is set of different types of player \( i \) (total of \( n \) sets of types)

Example - from previous page \( T_1 = \{1,2,3\}, T_2 = \{1,2,3,4\} \)

Example - 2 types of player 1, 2 types of player 2 and 3 types of player 3:

\[ T_1 = \{a,b\}, T_2 = \{c,d\}, T_3 = \{x,y,z\} \]

Specific Player - \( t_{ij} \) is the \( j^{th} \) type of player \( i \) (this was denoted \( t_i^j \) on previous page)

Other Player Types - \( T_{-i} = T_1 \times T_2 \times \ldots \times T_{i-1} \times T_{i+1} \times \ldots \times T_n \) the Cartesian product of the sets of all player types except for player \( i \)

Example - \( T_{-3} = \{(a,c), (a,d), (b,c), (b,d)\} \)

Other Players - \( t_{-i} \in T_{-i} \) is a specific type of each player (except player \( i \))

Example - one of the elements of \( T_{-3} \) above

Beliefs - \( p_{ij} = \Pr[t_{-i} | t_{ij}] \) = subjective belief of \( j^{th} \) type of player \( i \) about the probability that this specific combination of opponents \( t_{-i} \) will occur

\( p_{ij} \) = set of all \( p_{ij} \) for player \( i \)

Example - \( p_{3x} = (0.25, 0.25, 0.25, 0.25), p_{3y} = (0.2, 0.2, 0.3, 0.3), p_{3z} = (0.5, 0.5, 0, 0) \)

So type \( x \) of player 3 thinks it’s equally likely to see any combination of types of players 1 and 2; type \( y \) of player 3 thinks he’ll face \( (a,c) \) (i.e., type \( a \) of player 1 and type \( c \) of player 2) about a fifth of the time; etc.

\( p_3 = \{p_{3x}, p_{3y}, p_{3z}\} \)

Preferences - \( u_i(s,t) \); utility for player \( i \) depends on the actual types of each player (\( t \)) and the strategies that each player uses (\( s \))

Bayesian Game of Incomplete Information - \( \Gamma(S_1, \ldots, S_n, T_1, \ldots, T_n, p_1, \ldots, p_n, u_1, \ldots, u_n) \)

(assume this is common knowledge)

Example of Beliefs - in poker, there are \( 52!/(47!5!) \) possible hands; prior to dealing, all hands have equal probability; after seeing your cards (and others available) the probabilities assigned to different types of opponents (i.e., hands they have) change... for example, if you have 2 aces, the probability that an opponent has a three or four of a kind with aces or any other hand that requires more than 2 aces will be zero

Professional Poker - assume player 1 has 2 queens and player 2 has 2 jacks; on TV, they say in this scenario player 1 has a 90% chance of winning and player 2 only has a 10% chance, suggesting that it’s a bad idea for player 2 to stay in the game... but that analysis is based on complete information; the players have incomplete information so the relevant probabilities are each player’s (subjective) probability of winning given his
own cards (and those he's seen); that is, each player is basing his decision on his belief about his opponent's hand (usually a lot more probability on poor hands than good hands); if both players are still in the game, that means they both think they have greater than 50% chance of winning.

Consistency - \( p_{ij} = \frac{\text{Pr}[t_{-i} | t_{ij}]}{\text{Pr}(t_{ij})} \), where \( \text{Pr}(t_{ij}) = \sum_{t_{-i}} \text{Pr}[t_{-i}, t_{ij}] \)

Example - 3 players each with 2 types has total of 8 combinations (so each type of each player faces 4 combinations of possible opponents):

\[
\begin{align*}
& t_{11}, t_{21}, t_{31} \\
& t_{12}, t_{21}, t_{31} \\
& t_{11}, t_{22}, t_{31} \\
& t_{12}, t_{22}, t_{31} \\
& t_{11}, t_{21}, t_{32} \\
& t_{12}, t_{21}, t_{32} \\
& t_{11}, t_{22}, t_{32} \\
& t_{12}, t_{22}, t_{32}
\end{align*}
\]

No Big Deal - Almond showed inconsistent beliefs can be transformed into consistent beliefs; we usually just assume beliefs are consistent because of this.

Max Expected Utility - \( \max_{s_i} \sum_{t_{-i}} u_i(s_{ij}, s_{-i}(t_{-i}); t_{ij}, t_{-i}) \cdot \text{Pr}[t_{-i} | t_{ij}] \)

Suppose belief \( p_{ij} = \text{Pr}[t_{-i} | t_{ij}] \) is not consistent (\( u_i(s, t) \) above is incorrect)

Suppose belief \( \hat{p}_{ij} \) is consistent

Always Consistent - let \( N(T_{-i}) = \# \) of vectors \( t_{-i} \in T_{-i} = \) product of number of types of all players except \( i \); (example above has \( N(T_{-i}) = 4 \))

Assume each \( t_{-i} \) is equally likely so \( \hat{p}_{ij} = \text{Pr}[t_{-i} | t_{ij}] = 1 / N(T_{-i}) \)

Based on the definition of consistency, belief \( \hat{p}_{ij} \) is consistent

Define new utility \( \hat{u}_i(s, t) = N(T_{-i}) \cdot p_{ij} \cdot u_i(s, t) \)

Note that \( \hat{p}_{ij} \cdot \hat{u}_i(s, t) = p_{ij} \cdot u_i(s, t) \)

That means expected utility with consistent and inconsistent beliefs is the same

Catch - have to know the "weird" (inconsistent) probabilities to incorporate into preferences (new utility function)
**Equilibrium** - two ideas on how to find it

**Players Approach** - treat each type of each player as a separate player (e.g., 2 players, each with 2 types becomes a 4 player game); "players approach" is Len's term; Slutsky called it the **ex post** approach (meaning players choose their strategies after knowing their type)

**Game Structure** - not all players will face each other (e.g., \( t_{11} \) doesn’t play against \( t_{12} \)); this means the reaction function only depends on some of the opponents (in the example it would be 2 opponents instead of 3)

**Interaction** - (another Poker aside) a player of one type may want to affect payoffs when he’s another type; Examples:

- **Betting** - bet with a bad hand so opponent knows you bet with a bad hand and then doesn’t know when you have a good hand

- **Bluffing** - if you bluff and the opponent folds and asks to see your cards there are two options:
  - Tell opponent that to see the hand, he needs to call (match bet)... in other words, he has to pay for the information
  - Show him your hand so he knows you’re bluffing; that way if you bet latter with a good hand, he might think you’re bluffing and bet against you

**Problem** - these are use repeated game reasoning (we’re still focusing on single-shot games)

\[
\sigma(t_j) \text{ is an equilibrium strategy for } t_j \text{ (type } j \text{ of player } i) \text{ if } \forall \: i \text{ and } t_j \in T_i,
\sum_{t_i \in T_i} u_i(\sigma(t_j),\sigma_{-i};t_j,t_{-i}) \cdot \Pr[t_{-i} | t_j] \geq \sum_{t_i \in T_i} u_i(s_i,\sigma_{-i};t_j,t_{-i}) \cdot \Pr[t_{-i} | t_j] \forall \: s_i \in S_i
\]

**English** - taking the other players’ strategies \((\sigma_{-i})\) as given, player \( t_{ij} \)'s best reply is \( \sigma(t_j) \) (i.e., the expected payoff is greater than or equal to the expected payoff of playing any other strategy \( s_i \) in his strategy space \( S_i \)); in order for this to be an equilibrium, all the players must be playing best replies to their opponents' best replies so this condition holds for all types of all players (\( \forall \: i \text{ and } t_j \in T_i \))

**Note:** alternative strategy doesn’t need to be indexed by the player type (\( j \)) because we assumed that all types of the same player have the same strategy space (top of p.2)

**Strategies Approach** - first move in game is nature choosing players types; solve an imperfect, but complete information game; "strategies approach" is Len's term; Slutsky called it the **ex ante** approach (players choose their strategies before knowing their type)

- **+**: Fewer players
- **-**: More complicated strategies (must address strategies for each player type)

**Different?** - many questions on whether these two methods are different in equilibria or method (e.g., easier to solve)

**Subgame Perfection** - the players approach guarantees subgame perfect; some people argue the strategies approach could have strategies that aren’t subgame perfect

**Harsanyi** - argued the two were different; idea of immediate vs. delayed commitment (here, "commitment" means picking a strategy)

**Immediate Commitment** - at start of game before knowing type (strategies approach)
**Delayed Commitment** - choose strategy after knowing type (players approach)

Slutsky - "The terminology is no longer used, in part because it's not useful."

**Problem** - Harsanyi argued these are different and said delayed commitment is better, but his example problem combined cooperative and non-cooperative elements

Slutsky - not sure what equilibrium notion is in a mixed cooperative and non-cooperative game

**Same** - in fully non-cooperative game strategies and players approach are THE SAME; some fields (e.g., regulation) use the players approach because "it's right", but they're incorrectly apply Harsanyi conclusion to fully non-cooperative games

**History** - Blackwell, Nash, Von-Neumann, Kuhn all did two-player poker examples using the strategies approach 15 years before Harsanyi; they usually used a simplified version like 2 players each with 3 types (high, medium, and low hand); in that case, strategies approach is easier then players approach (2 player, 8 strategy game vs. 6 player, 2 strategy)

**Players Approach Easier** - for infinite types with continuous strategy spaces, the "easy" way is to parameterize type and solve using the players approach (e.g., principal agent problem; consumer maximization; firm profit maximization)

**Example** - simples case: 2 players, 2 types of each, 2 strategies: \( a_1, a_2 \) and \( b_1, b_2 \)

**Players Approach** - game shown here is for player \( t_{11} \) (only shows his payoffs even though we need to know player 2's to solve the game); at a minimum, we also need another pair of payoff tables for player \( t_{12} \); that will contain all the information although Slutsky prefers to have a pair for each player type (total of four of these)

Notation:

\[
\pi_{11} = \Pr[t_{11} \text{ plays } a_1] \quad (\text{so } 1 - \pi_{11} = \Pr[t_{11} \text{ plays } a_2])
\]

\[
p_{11} = \Pr[t_{11} \text{ believes } t_{21} \text{ is his opponent}] \quad (\text{so } 1 - p_{11} = \Pr[t_{11} \text{ believes } t_{22} \text{ is his opponent}])
\]

So the probabilities always default to the first strategy or the first type of the opponent

**Expected Utility** -

\[
E[u_{11}] = p_{11}\pi_{21}x_{11} \cdot x_{1} + p_{11}(1 - \pi_{21})\pi_{21} \cdot x_{2} + p_{11}\pi_{22}(1 - \pi_{11}) \cdot x_{3} + p_{11}(1 - \pi_{22})(1 - \pi_{11}) \cdot x_{4} + \]

\[
(1 - p_{11})\pi_{22}x_{12} \cdot y_{1} + (1 - p_{11})(1 - \pi_{22})\pi_{22} \cdot y_{2} + (1 - p_{11})\pi_{22}(1 - \pi_{11}) \cdot y_{3} + \]

\[
(1 - p_{11})(1 - \pi_{22})(1 - \pi_{11}) \cdot y_{4}
\]

This is linear in \( \pi_{11} \), so we can write \( t_{11} \) problem as

\[
\max_{\pi_{11}} \pi_{11}f(p_{11}, x_{1} - x_{4}, y_{1} - y_{4}, \pi_{21}, \pi_{22}) + c ... \quad \text{only } \pi_{21} \text{ & } \pi_{22} \text{ are from other players}
\]

**Best Reply** -

\[
\pi_{11} = \begin{cases} 
1 & \text{if } f(\bullet) > 0 \\
0 & \text{if } f(\bullet) < 0 \\
\in (0,1) & \text{if } f(\bullet) = 0 
\end{cases}
\]

There will be a best reply for the other three player types resulting in 4 equations with 4 unknowns

\( (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) \)
Strategies Approach - will show this is the same as the players approach; assume Eq is the pure strategy equilibrium

New Notation - \( p_{11} = \Pr[t_{11}] \) (probability player 1 is type 1)

Same as Players Approach - use expected payoffs to show \( ex \ post \Leftrightarrow ex \ ante \)

Look at Eq first:

Player 1 type 1 (prob is \( p_{11} \)) plays \( a_2 \); his opponent could be type 1 of player 2 (prob \( \Pr[t_{21}] \)) and plays \( b_2 \) or type 2 of player 2 (prob \( \Pr[t_{22}] \)) and plays \( b_1 \)

Player 1 type 2 (prob is \( p_{12} \)) plays \( a_1 \); his opponents are the same and play the same strategies (but the conditional probabilities are different)

\[
E[Eq] = p_{11} \left[ \Pr[t_{21} \mid t_{11}] u^1(a_2, b_2; t_{11}, t_{21}) + \Pr[t_{22} \mid t_{11}] u^1(a_1, b_1; t_{11}, t_{22}) \right] \\
+ p_{12} \left[ \Pr[t_{21} \mid t_{12}] u^1(a_2, b_2; t_{12}, t_{21}) + \Pr[t_{22} \mid t_{12}] u^1(a_1, b_1; t_{12}, t_{22}) \right]
\]

Since this is an equilibrium, we know: \( E[Eq] \geq E[c4] \)

\( c4 \) has the following strategies \( (a_2, a_2, b_1, b_1) \) so basically only the strategy for player \( t_{12} \) is changed (from \( a_1 \) to \( a_2 \)); using the same logic as above we have

\[
E[r1] = p_{11} \left[ \Pr[t_{21} \mid t_{11}] u^1(a_2, b_2; t_{11}, t_{21}) + \Pr[t_{22} \mid t_{11}] u^1(a_2, b_1; t_{11}, t_{22}) \right] \\
+ p_{12} \left[ \Pr[t_{21} \mid t_{12}] u^1(a_2, b_2; t_{12}, t_{21}) + \Pr[t_{22} \mid t_{12}] u^1(a_1, b_1; t_{12}, t_{22}) \right]
\]

The corresponding circled terms cancel out so \( E[Eq] \geq E[c4] \) becomes

\[
p_{12} \left[ \Pr[t_{21} \mid t_{12}] u^1(a_1, b_2; t_{12}, t_{21}) + \Pr[t_{22} \mid t_{12}] u^1(a_1, b_1; t_{12}, t_{22}) \right] \geq \\
p_{12} \left[ \Pr[t_{21} \mid t_{12}] u^1(a_2, b_2; t_{12}, t_{21}) + \Pr[t_{22} \mid t_{12}] u^1(a_2, b_1; t_{12}, t_{22}) \right]
\]

That is, the expected payoff of \( a_1 \) once player 1 knows he's type 2 is \( \geq \) to the expected payoff of \( a_2 \) once he knows he's type 2: \( E[u^{12}(a_1)] \geq E[u^{12}(a_2)] \)

We can use this same logic to get

\( E[Eq] \geq E[c1] \) yields same condition as \( t_{11} \) playing \( a_2 \) over \( a_1 \)

\( E[Eq] \geq E[r1] \) yields same condition as \( t_{21} \) playing \( b_2 \) over \( b_1 \)

\( E[Eq] \geq E[r4] \) yields same condition as \( t_{22} \) playing \( b_1 \) over \( b_2 \)

Combining all four comparison yields the same equilibrium strategy as the \( ex \ post \) game:

\[
s_{11} = a_2, \ s_{12} = a_1, \ s_{21} = b_2, \ s_{22} = b_1
\]

\( \therefore \) equilibrium in \( ex \ ante \) game is the same equilibrium in the \( ex \ post \) game

Now assume \( s_{11} = a_2, \ s_{12} = a_1, \ s_{21} = b_2, \ s_{22} = b_1 \) is an equilibrium in the \( ex \ post \) game; we want to show this is also an equilibrium in the \( ex \ ante \) game; using the work above, we can work backwards to show \( Eq = (a_2, a_1, b_2, b_1) \) has a higher expected payoff than \( r1, r4, c1, c4 \); now consider \( E[Eq] \) vs. \( E[r2] \); i.e., compare

\[
(s_{11} = a_2, \ s_{12} = a_1, \ s_{21} = b_2, \ s_{22} = b_1) \text{ to } (s_{11} = a_1, \ s_{12} = a_2, \ s_{21} = b_2, \ s_{22} = b_1) \text{ so both types of player 1 are changing their strategy; changing the notation a little for convenience:}
\]

\[
E[Eq] = p_{11} E[u^{11}(a_2)] + p_{12} E[u^{12}(a_1)] \\
E[r2] = p_{11} E[u^{11}(a_1)] + p_{12} E[u^{12}(a_2)]
\]

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From the equilibrium assumption, we know $p_{11} E[u^{11}(a_2)] \geq p_{11} E[u^{11}(a_1)]$ and $p_{12} E[u^{12}(a_1)] \geq p_{12} E[u^{12}(a_2)]$ (i.e., each term in $E[Eq]$ is $\geq$ it's respective term in $E[r2]$)... that means $E[Eq] \geq E[r2]$

We can repeat this logic to show $E[Eq] \geq E[c2] \therefore (s_{11} = a_2, s_{12} = a_1, s_{21} = b_2, s_{22} = b_1)$

in the $ex \ post$ game is the same as $(a_2a_1, b_2b_1)$ in the $ex \ ante$ game

**Why?** - this works because the types are independent of each other; we can't have $t_{11}$ and $t_{12}$ at the same time so there is no interaction (i.e., can change both their strategies at the same time and it's equivalent to change one at a time)

**Mixed Strategies** - first look at $ex \ post$; suppose $t_{11}$ plays $(1/2, 1/2)$ and $t_{12}$ plays $(1/3, 2/3)$, in the $ex \ ante$ game, that's the same as:

<table>
<thead>
<tr>
<th></th>
<th>$t_{11}$</th>
<th>$t_{12}$</th>
<th>$ex \ ante$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1, a_1$</td>
<td>1/3</td>
<td>1/6</td>
<td></td>
</tr>
<tr>
<td>$a_1, a_2$</td>
<td>1/2</td>
<td>2/3</td>
<td>2/6</td>
</tr>
<tr>
<td>$a_2, a_1$</td>
<td>1/2</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>$a_2, a_2$</td>
<td>2/3</td>
<td>2/6</td>
<td></td>
</tr>
</tbody>
</table>

**Problem?** - going the other way, it's possible to come up with a mixed strategy in the $ex \ ante$ game that can't be replicated by mixed strategies in the $ex \ post$ game (e.g., $(0, 1/2, 1/2, 0)$); this is part of the intuition why Harsanyi said they were different

**No Problem** - won't get this as a mixed strategy because there's no gain in correlation between the type types; $(0, 1/2, 1/2, 0)$ might as well be $(1/4, 1/4, 1/4, 1/4)$ which also has each type playing a 50-50 mixed strategy; in a real 4 player game there could be correlated strategies so this isn't the case