Risk Aversion (Kreps Chpt 6)

What is Z - typically, we'll assume $Z = (\underline{z}, \overline{z}) \subseteq R$ (i.e., an interval of the real line); each value $z \in Z$ can represent cash or the amount of some commodity

Notation -

P_S - set of simple probability distributions on Z (described on previous page)

$\mathbf{p}, \mathbf{q}, \mathbf{r}$ - typical elements of P_S

$\succ$ - binary relation denoting preferences over P_S (i.e., $\succ \subseteq P_S \times P_S$

$\overline{p}_{\hat{z}}$ or $\delta_{\hat{z}}$ - degenerate lottery at the value $\hat{z}$ (i.e., $p(z) = 1$ if $z = \hat{z}$, 0 otherwise); all mass at one point

$E[f \mid \mathbf{p}]$ - expected value of function f taken with respect to $\mathbf{p}$

$E[\mathbf{p}]$ - expected value of $\mathbf{p}$

$v[\mathbf{p}]$ - variance of $\mathbf{p}$

KP 6.1 - $u : Z \to R$ is <u>strictly increasing</u> *iff* $\overline{p}_z \succ \overline{p}_{z'} \Leftrightarrow z > z'$

Risk Aversion - preferences $\succ$ are

Risk Averse if $\overline{p}_{E[z|\mathbf{p}]} \succeq \mathbf{p} \;\; \forall \; \mathbf{p} \in P_S$ (i.e., weakly prefer mean for certain over gamble)

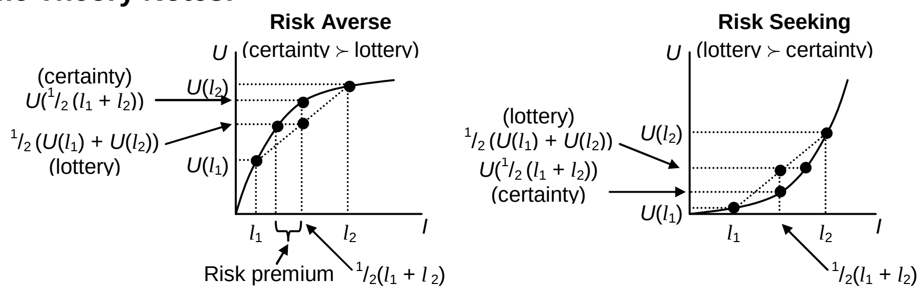
Strictly Risk Averse if $\overline{p}_{E[z|\mathbf{p}]} \succ \mathbf{p} \;\; \forall \; \mathbf{p} \in P_S$ such that $v[\mathbf{p}] > 0$

Risk Neutrality if $\overline{p}_{E[z|\mathbf{p}]} \sim \mathbf{p} \;\; \forall \; \mathbf{p} \in P_S$ (i.e., indifferent between mean for certain or gamble)

Risk Seeking if $\mathbf{p} \succeq \overline{p}_{E[z|\mathbf{p}]} \;\; \forall \; \mathbf{p} \in P_S$ (i.e., weakly prefer gamble over mean for certain)

Strictly Risk Seeking if $\mathbf{p} \succ \overline{p}_{E[z|\mathbf{p}]} \;\; \forall \; \mathbf{p} \in P_S$ such that $v[\mathbf{p}] > 0$

From Game Theory Notes:



Definitions & Propositions-

Concave - function $f : Z \to R$ is concave if $f(\alpha z + (1-\alpha)y) \geq \alpha f(z) + (1-\alpha) f(y) \;\; \forall$ $y, z \in Z$ and $\alpha \in (0,1)$

Jensen's Inequality - For $\mathbf{p}$ a simple probability distribution on Z, $f(e[\mathbf{p}]) \geq E[f \mid \mathbf{p}]$ if f is concave; strict inequality if strictly concave; equality if f is affine;

Continuity - concave function u is define on an open interval, u is continuous and continuously differentiable almost everywhere (if not differentiable, take limit of u' from left and right to be left and right hand derivatives)

u' is nonincreasing (i.e., $u'' \leq 0$)

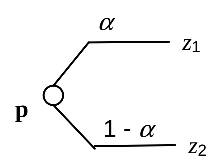
Convex - reverse the inequalities above

KP 6.2 - preferences $\succ$ are

- **Risk Averse** $\Leftrightarrow$ u is concave
- **Strictly Risk** $\Leftrightarrow$ u is strictly concave
- **Risk Neutral** $\Leftrightarrow$ u is affine
- **Risk Seeking** $\Leftrightarrow$ u is convex
- **Strictly Risk Seeking** $\Leftrightarrow$ u is strictly convex

Proof: "partial sketch"

Risk Averse $\Rightarrow$ concave:



Let $\mathbf{p} = \alpha\overline{p}_{z_1} + (1-\alpha)\overline{p}_{z_2}$, $\alpha \in (0,1)$

Risk aversion $\Rightarrow$ $\overline{p}_{E[z|\mathbf{p}]} \succeq \mathbf{p}$

Use KP 5.15: $\overline{p}_{E[z|\mathbf{p}]} \succeq \mathbf{p} \Leftrightarrow E[u \mid \overline{p}_{E[z|\mathbf{p}]}] \ge E[u \mid \mathbf{p}]$

Now use $E[u \mid \overline{p}_{E[z|\mathbf{p}]}] = E[u \mid \alpha z_1 + (1-\alpha) z_2]$ and $E[u \mid \mathbf{p}] = \alpha u(z_1) + (1-\alpha) u(z_2)$

$u(\alpha z_1 + (1-\alpha) z_2) \ge \alpha u(z_1) + (1-\alpha) u(z_2)$ so u is concave

i.e., the function of the average value of z is $\ge$ the average of the function of each z

For lottery, that means, utility of expected payoff of lottery is $\ge$ expected utility of lottery

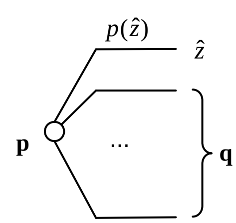
Concave $\Rightarrow$ risk averse

Assume u is concave

Since we have finite support, we can use induction

Case $n = 1$ is trivial (vacuously true because we can't compare two things when we only have one)

Assume concave for $n-1$



$\mathbf{p} = p(\hat{z})\overline{P}_{\hat{z}} + (1 - p(\hat{z}))\mathbf{q}$, where $\mathbf{q} = \dfrac{p(z)}{1 - p(\hat{z})}$ if $z \ne \hat{z}$, 0 otherwise

$E[u \mid \mathbf{p}] = p(\hat{z})u(\hat{z}) + (1 - p(\hat{z}))E[u \mid \mathbf{q}]$

Use induction hypothesis: $E[u \mid \mathbf{q}] \le E[u \mid \overline{q}_{E[z|\mathbf{q}]}]$

$\therefore$ $E[u \mid \mathbf{p}] \le p(\hat{z})u(\hat{z}) + (1 - p(\hat{z}))E[u \mid \overline{q}_{E[z|\mathbf{q}]}]$

By definition $E[u \mid \overline{q}_{E[u|\mathbf{q}]}] = u(E[z \mid \mathbf{q}])$ ... $\overline{q}_{E[u|\mathbf{q}]}$ has a single value: $E[z \mid \mathbf{q}]$

$\therefore E[u \mid \mathbf{p}] \le p(\hat{z})u(\hat{z}) + (1 - p(\hat{z}))u(E[z \mid \mathbf{q}])$

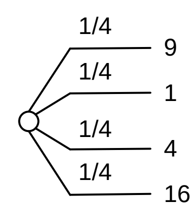
This is a convex combination of $u(\hat{z})$ and $u(E[z \mid \mathbf{q}])$ $\therefore$ since we assumed u is concave, we know $p(\hat{z})u(\hat{z}) + (1 - p(\hat{z}))u(E[z \mid \mathbf{q}]) \le u(E[z \mid \mathbf{p}])$

$\therefore$ $E[u \mid \mathbf{p}] \le u(E[z \mid \mathbf{p}])$

By definition $E[u \mid \overline{p}_{E[u|\mathbf{p}]}] = u(E[z \mid \mathbf{p}])$ so we have $E[u \mid \mathbf{p}] \le E[u \mid \overline{p}_{E[z|\mathbf{p}]}]$

By KP 5.15 that means $\mathbf{p} \preceq \overline{p}_{E[z|\mathbf{p}]}$ which means preferences are risk averse for n lotteries

Example -

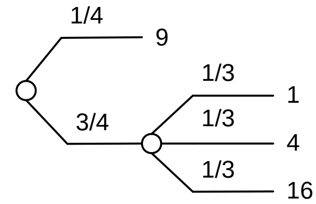


Let's use $u = \sqrt{z}$, which is a convex function. Look at the full game with $n = 4$ alternatives. It's easy to verify risk aversion from the full tree:

$E[z \mid \mathbf{p}] = \frac{1}{4}(9 + 1 + 4 + 16) = 7.5 \therefore u(E[z \mid \mathbf{p}]) = E[u \mid \overline{p}_{E[u|\mathbf{p}]}] = \sqrt{7.5} \approx 2.7$

$E[u \mid \mathbf{p}] = \frac{1}{4}(3+1+2+4) = 2.5$

$2.7 > 2.5 \Rightarrow E[u \mid \overline{p}_{E[u|\mathbf{p}]}] > E[u \mid \mathbf{p}] \Rightarrow \overline{p}_{E[u|\mathbf{p}]} \succ \mathbf{p} \Rightarrow$ risk averse



To follow the induction the proof, though, we want to look at breaking the tree into $n = 3$ and adding another branch

$E[u \mid \mathbf{p}] = \frac{1}{4}(3) + \frac{3}{4}E[u \mid \mathbf{q}]$

The induction hypothesis says $E[u \mid \mathbf{q}] \le E[u \mid \overline{q}_{E[z|\mathbf{q}]}]$, which we can verify with the numbers:

$E[u \mid \mathbf{q}] = \frac{1}{3}(1+2+4) = \frac{7}{3} \approx 2.3$

$E[u \mid \overline{q}_{E[z|\mathbf{q}]}] = \sqrt{\frac{1}{3}(1+4+16)} = \sqrt{7} \approx 2.6$

$\therefore$ we can write $E[u \mid \mathbf{p}] \le \frac{1}{4}(3) + \frac{3}{4}E[u \mid \overline{q}_{E[z|\mathbf{q}]}]$

At this point, the proof uses the fact that $E[u \mid \overline{q}_{E[u|\mathbf{q}]}] = u(E[z \mid \mathbf{q}])$, which we can verify with the numbers:

$u(E[z \mid \mathbf{q}] = \sqrt{\frac{1}{3}(1+4+16)} = \sqrt{7} \approx 2.6$

So the proof rewrites things as $E[u \mid \mathbf{p}] \le \frac{1}{4}(3) + \frac{3}{4}u(E[z \mid \mathbf{q}])$

Now because of the convexity of u, the proof claims

$\frac{1}{4}(3) + \frac{3}{4}u(E[z \mid \mathbf{q}]) \le u(E[z \mid \mathbf{p}])$ ... to the numbers we go:

$\frac{1}{4}(3) + \frac{3}{4}u(E[z \mid \mathbf{q}]) = \frac{1}{4}(3) + \frac{3}{4}\sqrt{7} \approx 2.734$

$u(E[z \mid \mathbf{p}]) = \sqrt{7.5} \approx 2.738$ (OK, technically, we used the whole tree to figure this out, but I'm just trying to use the numbers to clarify what's going on in the proof)

Put the latest step in and we get $E[u \mid \mathbf{p}] \le u(E[z \mid \mathbf{p}])$

Invoke the definition: $u(E[z \mid \mathbf{p}]) = E[u \mid \overline{p}_{E[u|\mathbf{p}]}]$ and we have $E[u \mid \mathbf{p}] \le E[u \mid \overline{p}_{E[z|\mathbf{p}]}]$

KP 5.15 finishes off the proof by saying $\mathbf{p} \preceq \overline{p}_{E[z|\mathbf{p}]}$ which means preferences are risk averse for the case $n = 4$

Concavity Meaningful? - in first problem set, we said concavity is not meaningful property for utility representations, how can we then use it to determine risk aversion?

Before - we were looking at ordinal measurement so we allowed strictly increasing transformations

Now - admissible transformations are positive affine ($a + bu(z)$ with $b > 0$); if we know $u'' > 0$ then $bu'' > 0$ so if u is concave, then $a + bu(z)$ will also be concave

Certainty Equivalents (Kreps Chpt 6, p.73)

Certainty Equivalent - CE($\mathbf{p}$) $\equiv \{z \in Z : \bar{p}_z \sim \mathbf{p}\}$... i.e., certain amount that is indifferent to the gamble (or lottery)

KP 6.3 - if $u(\cdot)$ is strictly increasing and concave (i.e., risk averse) $\Rightarrow$ |CE($\mathbf{p}$)| = 1

English - cardinality of CE($\mathbf{p}$) is 1 (i.e., there's only one certainty equivalent for gamble $\mathbf{p}$)

Other Definition - $E[u \mid \mathbf{p}] = u(\text{CE}(\mathbf{p}))$

Note1: this is how we calculate the CE (i.e., find the expected utility and then determine the certain payoff that yields that same level of utility)

Note2: this definition is used in proof of CP 5.1

New Interpretation of Preference - $\mathbf{p} \succ \mathbf{q} \Leftrightarrow \text{CE}(\mathbf{p}) > \text{CE}(\mathbf{q})$

Buying vs. Selling Prices - from problem set 1: $u(x) = \sqrt{200 + x}$ ... increasing and concave

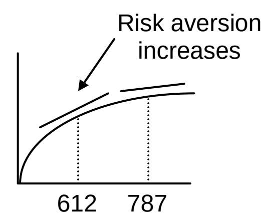
CE w/out info is 611.76

CE w/ info is 787.20

$\therefore$ if the person owns the info, it's worth 787.20

If person doesn't have info, how much is he willing to pay for it?

Note, pulling payment out, increases risk aversion so we can't just subtract 611.76 from 787.20 to determine what information is worth (can only do difference like that if there's **constant absolute risk aversion**... see p.7)



Actual value of info is 166.74... see problem set 1, problem 1

Odd Result - note from PS1 that CE with risk neutral > CE for risk averse, but risk averse person is willing to pay more for the information

Risk Premium

Risk Averse - $\pi(\mathbf{p}) = E[z \mid \mathbf{p}] - \text{CE}(\mathbf{p}) > 0$ (for strictly increasing and concave $u(\cdot)$)

Risk Neutral - $\pi(\mathbf{p}) = 0$

Arrow-Pratt

Defined for $u' > 0$ and $u'' < 0$

$$A(z) \equiv \frac{-u''(z)}{u'(z)} = -\frac{d}{dz}\ln u'(z) > 0$$ "change in the change, normalized by the change"

Meaningful? - get same measure with positive affine transformation so it is meaningful

Proof:

$$-\int A(z)dz = B + \ln u'(z)$$

$$\exp\left[-\int A(z)dz\right] = e^B u'(z) = bu'(z)$$

$$\int \exp\left[-\int A(z)dz\right] = a + bu(z)$$

Fancy Math - with $\mathbf{p}$ is "small" and $E[z \mid \mathbf{p}] = 0$ ... I have no clue what this means, but it's an "intuitive description of the Arrow-Pratt measure"

$\mathbf{p} \circ \hat{z}$ is concatenation: $\mathbf{p} \circ \hat{z}(z) = \mathbf{p}(z - \hat{z})$

$$E[u \mid \mathbf{p} \circ \hat{z}] = E\{u(\hat{z}) + (z - \hat{z})u'(\hat{z}) + \tfrac{1}{2}(z - \hat{z})u''(\hat{z}) + R\} \cong u(\hat{z}) + \tfrac{1}{2}u''(\hat{z})v(\mathbf{p})$$

$$u(\text{CE}) = u(\hat{z} - \pi(\mathbf{p} \circ \hat{z})) \cong u(\hat{z}) - \pi(p \circ \hat{z})u'(\hat{z}) + \tfrac{1}{2}\pi(p \circ \hat{z})^2 u''(\hat{z})$$ assumed "small"

Equating: $\frac{1}{2}u''(\hat{z})v(\mathbf{p}) = -\boldsymbol{\pi}(p \circ \hat{z})u'(\hat{z})$

$$\frac{\boldsymbol{\pi}(p \circ \hat{z})}{\frac{1}{2}v(\mathbf{p})} = -\frac{u''(\hat{z})}{u'(\hat{z})} = A(\hat{z}) \quad \text{or} \quad \boldsymbol{\pi}(p \circ \hat{z}) = \tfrac{1}{2}A(\hat{z})v(\mathbf{p})$$

(Class Proposition) **CP 5.1** - The following are equivalent:

(i) $A_2(z) \geq A_1(z) \ \forall \ z \in Z$ ... i.e., larger Arrow-Pratt measure

(ii) $\exists$ concave transformation $G: R \to R$, $G' > 0$, $G'' < 0$ such that $u_2 = G(x_1)$

Note: $G' > 0$ preserves order; $G'' < 0$ "bends it more"... i.e., more risk averse; class notes used $G \circ u^*$ which is the same as $G(u^*)$, but just one more way to confuse us

(iii) $\boldsymbol{\pi}_2(\mathbf{p}) \geq \boldsymbol{\pi}_1(\mathbf{p}) \ \forall \ \mathbf{p} \in X$ ... i.e., larger risk premium

Note: changed from class notes; used u_1 for u^* and $u_2 = G(u_1)$ for $u = G \circ u^*$; this makes the notation much easier to follow because the * doesn't get in the way... just think of u_2 being "more risk averse" than u_1

English - more concave utility ⇔ more risk averse ⇔ larger Arrow-Pratt ⇔ larger risk premium ⇔ smaller CE

Proof:

(i) ⇒ (ii)

Assume u and u^* are strictly increasing and $\exists$ G such that $u = G(u^*)$

$u' = G' \cdot u^{*\prime}$... by assumption $u' > 0$ and $u^{*\prime} > 0$ so we must have $G' > 0$

$u'' = G'' \cdot (u^{*\prime})^2 + G' \cdot u^{*\prime\prime}$

Solve for G'': $G'' = \dfrac{u'' - G' \cdot u^{*\prime\prime}}{(u^{*\prime})^2}$

Not entirely intuitive step, but factor out $\dfrac{G'}{u^{*\prime}}$: $G'' = \dfrac{G'}{u^{*\prime}}\left[\dfrac{u''}{G \cdot u^{*\prime}} - \dfrac{u^{*\prime\prime}}{u^{*\prime}}\right]$

Substitute $u' = G' \cdot u^{*\prime}$: $G'' = \dfrac{G'}{u^{*\prime}}\left[\dfrac{u''}{u'} - \dfrac{u^{*\prime\prime}}{u^{*\prime}}\right]$

Substitute Arrow-Pratt measures: $G'' = \dfrac{G'}{u^{*\prime}}[A - A^*]$

From (i), we assumed $A \geq A^*$ so $A - A^* \leq 0$

We showed $G' > 0$ and assumed $u^{*\prime} > 0$

$\therefore G'' = \dfrac{G'}{u^{*\prime}}[A - A^*] = \dfrac{>0}{>0}[\leq 0]$ is ≤ 0, so G is concave transformation

(ii) ⇒ (iii)... here notation gets confusing so use u_1 and $u_2 = G(u_1)$

$u_2(\mathrm{CE}_2(\mathbf{p})) = E[u_2 \mid \mathbf{p}] = E[G(u_1) \mid \mathbf{p}]$ (using (ii))

Because G is concave: $E[G(u_1) \mid \mathbf{p}] \leq G(E[u_1 \mid \mathbf{p}])$

Substitute $E[u_1 \mid \mathbf{p}] = u_1(\mathrm{CE}_1(\mathbf{p}))$: $G(E[G(u_1) \mid \mathbf{p}]) = G(u_1(\mathrm{CE}_1(\mathbf{p})))$

Substitute $G(u_1(\mathrm{CE}_1(\mathbf{p}))) = u_2(\mathrm{CE}_1(\mathbf{p}))$

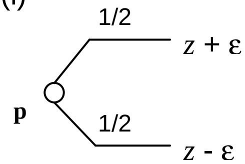
$\therefore \ u_2(\mathrm{CE}_2(\mathbf{p})) \leq u_2(\mathrm{CE}_1(\mathbf{p}))$

Since u_2 is an increasing function, $\mathrm{CE}_2(\mathbf{p}) \leq \mathrm{CE}_1(\mathbf{p})$

Combine that with definition of risk premium: $\boldsymbol{\pi}(\mathbf{p}) = E[z \mid \mathbf{p}] - \mathrm{CE}(\mathbf{p})$

$\therefore\ \boldsymbol{\pi}_2(\mathbf{p}) \geq \boldsymbol{\pi}_1(\mathbf{p})$

(iii) $\Rightarrow$ (i)



Using "Fancy Math" from bottom of p.3...

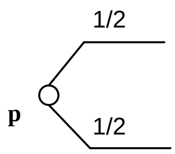
as $\boldsymbol{\varepsilon} \to 0$ $\boldsymbol{\pi}_1(\mathbf{p}) = \frac{1}{2} A_1(z)\boldsymbol{\varepsilon}^2$ and $\boldsymbol{\pi}_2(\mathbf{p}) = \frac{1}{2} A_2(z)\boldsymbol{\varepsilon}^2$

(iii) says $\boldsymbol{\pi}_2(\mathbf{p}) \geq \boldsymbol{\pi}_1(\mathbf{p})$

$\therefore\ A_2(z) \geq A_1(z)$

Example - this should make proof of CP 5.1 more clear

$u_1(z) = \sqrt{z}$; $u_2(z) = \sqrt{\sqrt{z}}$ (so $G(u_1) = \sqrt{u_1}$)



z	$u_1(z) = \sqrt{z}$	$u_2(z) = z^{1/4}$
0	0	0
100	10	3.162

(ii) says G is concave transformation... $\sqrt{\ }$ is a concave transformation

Arrow-Pratt:

$$u_1' = \tfrac{1}{2} z^{-1/2};\ u_1'' = -\tfrac{1}{4} z^{-3/2};\ A_1 = -\frac{-\frac{1}{4} z^{-3/2}}{\frac{1}{2} z^{-1/2}} = \tfrac{1}{2} z^{-1}$$

$$u_2' = \tfrac{1}{4} z^{-3/4};\ u_2'' = -\tfrac{3}{16} z^{-7/4};\ A_2 = -\frac{-\frac{3}{16} z^{-7/4}}{\frac{1}{4} z^{-3/4}} = \tfrac{3}{4} z^{-1}$$

(i) says $A_2(z) \geq A_1(z)$ ... in this case 3/4 > 1/2

Certainty Equivalents:

$$u_1(\mathrm{CE}_1) = \sqrt{\mathrm{CE}_1} = 5 \ \Rightarrow\ \mathrm{CE}_1 = 5^2 = 25$$

$$u_2(\mathrm{CE}_2) = (\mathrm{CE}_1)^{1/4} = \tfrac{1}{2}\sqrt{10} \ \Rightarrow\ \mathrm{CE}_2 = (\tfrac{1}{2}\sqrt{10})^4 = 6.25$$

As the proof argued $\mathrm{CE}_2(\mathbf{p}) \leq \mathrm{CE}_1(\mathbf{p})$ ... in this case 6.25 < 25

Risk Premiums:

$$\boldsymbol{\pi}_1 = E[z \mid \mathbf{p}] - \mathrm{CE}_1 = 50 - 25 = 25$$

$$\boldsymbol{\pi}_2 = E[z \mid \mathbf{p}] - \mathrm{CE}_2 = 50 - 6.25 = 43.75$$

(iii) says $\boldsymbol{\pi}_2(\mathbf{p}) \geq \boldsymbol{\pi}_1(\mathbf{p})$ ... in this case 43.75 > 25

Absolute Risk Aversion

$\forall$ (gambles) $\mathbf{p} \in X$ and (prizes) z_1, z_2, $z \in Z$ with $z_1 > z_2$ (specific prizes) (z is any prize)

Constant Absolute Risk Aversion (CARA) -

$$E(u \mid \mathbf{p} \circ z_2) > u(z_2 + z) \Leftrightarrow E(u \mid \mathbf{p} \circ z_1) > u(z_1 + z)$$

English - if given prize z_2 in wallet, person prefers gamble $\mathbf{p}$ over prize z, then person will prefer the gamble over z if he has z_1 in his wallet and vice versa

Plainer English - how much person has doesn't affect preferences

Decreasing Absolute Risk Aversion -

$$E(u \mid \mathbf{p} \circ z_2) > u(z_2 + z) \Rightarrow E(u \mid \mathbf{p} \circ z_1) > u(z_1 + z)$$

English - person becomes less risk averse as amount in wallet increases; the math above says if he prefers the game with a small amount (z_2), he'll prefer it with a large amount (z_1), but not necessarily the other way around

Increasing Absolute Risk Aversion -

$$E(u \mid \mathbf{p} \circ z_2) > u(z_2 + z) \Leftarrow E(u \mid \mathbf{p} \circ z_1) > u(z_1 + z)$$

English - person becomes more risk averse as amount in wallet increases; the math above says if he prefers the game with a large amount (z_1), he'll prefer it with a smaller amount (z_2), but not necessarily the other way around

CP 5.2 - relates absolute risk aversion to Arrow-Pratt measure

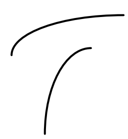
CARA $\Leftrightarrow$ $A(z)$ is constant... "constant concavity"

Example - $u = a - be^{-kz}$ ... $u' = bke^{-kz}$; $u'' = -bk^2e^{-kz}$; $A(z) = \dfrac{-u''}{u'} = -\dfrac{-bk^2e^{-kz}}{bke^{-kz}} = k$

Example - $u = a + bz$ ... $u' = b$; $u'' = 0$; $A(z) = \dfrac{-u''}{u'} = -\dfrac{0}{b} = 0$ (risk neutral)

Decreasing ARA $\Leftrightarrow$ $A(z)$ is decreasing (e.g., $\sqrt{\ }$)

Increasing ARA $\Leftrightarrow$ $A(z)$ is increasing

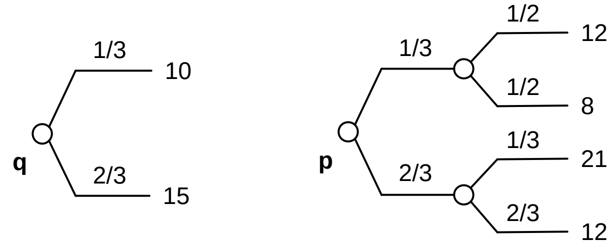


Risk Comparisons (Kreps p.89)

(intro from class handout 5)

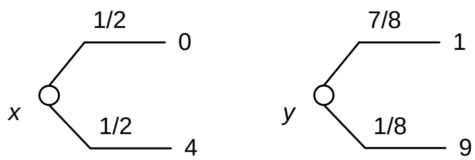
Is gamble **p** more or less risky than gamble **q** (for a given person)?

Assuming **p** and **q** have the same expected value, **p** is "more risky" if there is more uncertainty (i.e. "more weight in tails")... "as if" **p** is the same as **q** with extra noise



Point - this will eventually be applied to quality of information and the Blackwell Theorem

(from class handout 6; Ingersoll Handout)



Which gamble above is more risky?

Variance - rule of thumb says if they have the same mean, pick the gamble with the smallest variance... that means pick x

$$E(x) = \tfrac{1}{2}0 + \tfrac{1}{2}4 = 2$$

$$Var(x) = \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = \tfrac{1}{2}(0-2)^2 + \tfrac{1}{2}(4-2)^2 = 4$$

$$E(y) = \tfrac{7}{8}1 + \tfrac{1}{8}9 = 2$$

$$Var(y) = \tfrac{7}{8}(1-2)^2 + \tfrac{1}{8}(9-2)^2 = 7$$

Utility - suppose $u(z) = \sqrt{z}$ ... that means pick y

$$E(\sqrt{x}) = \tfrac{1}{2}0 + \tfrac{1}{2}2 = 1$$

$$E(\sqrt{y}) = \tfrac{7}{8}1 + \tfrac{1}{9}3 = \tfrac{10}{8} > 1$$

Trick Question - given 2 gambles with equal means; we want to say which is better (less risky), but as this example shows, we can't use variance and even utility is suspect because that means the answer is different for each person

Using Utility - if x and y are two random variables with $E(x) = E(y)$, we say y is **as risky as** x for all $u \in U$ if $E[u(x)] \geq E[u(y)]$ $\forall\ u \in U$

English - for the class of utility functions (e.g., "concave"), gamble x is less risky than gamble y if the expected utility of x is greater than for y; this is what's illustrated by second example above

Variations - "as risky as" = "more risky" = "riskier than"

General Assumption - focus on increasing, concave functions that are "sufficiently" differentiable (i.e., as many times as we need to take derivatives)

Alternatives - there are several ways to identify "riskiness" by focusing on the random variables themselves:

[1] $x \succeq y$ $\forall$ increasing u (note: doesn't require concave)

[2] $E[x] = E[y]$ and $x \succeq y$ $\forall$ increasing, concave u

[3] $E[x] = E[y]$ and $Var[x] \le Var[y]$

[4] $E[x] = E[y]$, $y \overset{d}{=} x + \varepsilon$ and $Cov(x, \varepsilon) = 0$

(i.e., y is equal in distribution to x plus noise <u>uncorrelated</u> with x)

[5] $E[x] = E[y]$, $y \overset{d}{=} x + \varepsilon$ and $E[\varepsilon \mid x] = 0$ (equal means is implied by last part)

[6] $E[x] = E[y]$, $y \overset{d}{=} x + \varepsilon$ and x & ε are <u>independent</u>

[7] $E[x] = E[y]$, y has "more weight in the tails" (meaning it's a "mean preserving spread")

Relationships -

[6] $\Rightarrow$ [5] $\Rightarrow$ [4]

[4] $\Leftrightarrow$ [3]

[2] $\Rightarrow$ [1]

[7] $\Leftrightarrow$ [5] $\Leftrightarrow$ [2]

Math Review -

Uncorrelated - means $Cov(x, \varepsilon) = 0$

Independent - means $Cov(f(x), g(\varepsilon)) = 0$ $\forall$ f and g

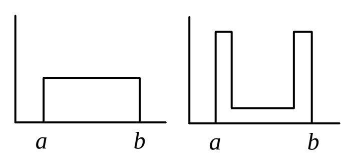
Conditionally Independent - means $E[x \mid y] = E[x]$ $\forall$ y; with some "regularity" this is equivalent to $Cov(x, g(y)) = 0$ $\forall$ g

Noise wrt y (Fair Game) - x conditionally independent of y with $E[x] = 0$ ... so $E[x \mid y] = 0$

Setup - x and y are random variables with respective cumulative distribution functions $F(x)$ and $G(y)$ on bounded support $[a, b]$... that is, $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$

Mean Preserving Spread -

Lower Tail - y has more weight in the lower tail if



$$\int_a^T (T - t)dF(t) \le \int_a^T (T - t)dG(t) \;\; \forall \; T \in [a, b]$$

Use integration by parts:

$\int UdV = UV - \int VdU$ ... let $U = (T - t)$ & $dV = dF(t)$ ... so $dU = -dt$ & $V = F(t)$

$$(T - t)F(t)\Big|_a^T + \int_a^T F(t)dt \le (T - t)G(t)\Big|_a^T + \int_a^T G(t)dt$$

At $t = T$, the first half of the first term on both sides drops out; at $t = a$, $F(a) = G(a) = 0$ so the second half of the first term on both sides drops out $\therefore$

$$\int_a^T F(t)dt \le \int_a^T G(t)dt$$

Upper Tail - y has more weight in the lower tails if $\int_T^b (T-t)dF(t) \leq \int_T^b (T-t)dG(t)$ $\forall$

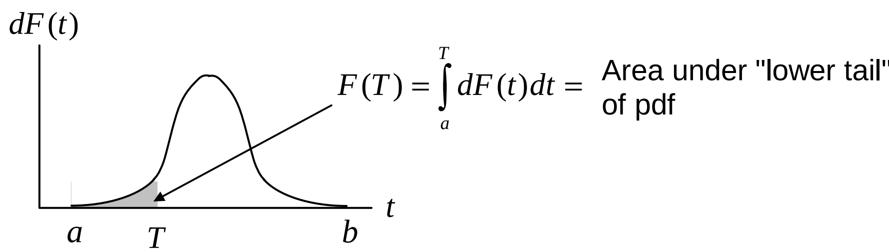
$T \in [a,b]$

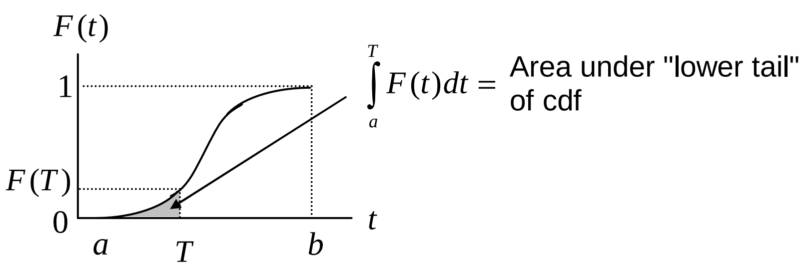
Use integration by parts as before to get :

$$\int_T^b F(t)dt \geq \int_T^b G(t)dt$$

Len's Note - this just seems like gratuitous calculus to confuse the crap out of us... think pictures!... also, if $F(T)$ is the CDF, then $dF(t)$ is the PDF

y has "more weight in lower tail"... in pdf, that means $F(T) \leq G(T)$ (i.e., the area below the pdf from T downward is less for the distribution of x than it is for y... the fancy math above says that the area is also less when looking at the CDF)





CP 6.1 - x and y are random variables with support $\subseteq [a,b]$ and $E[x] = E[y]$,

y has more weight in the lower tail *iff* y has more weight in the upper tail

Proof: more gratuitous calculus... aren't we smart?

$$E[X] = \int_a^b tdF(t) = tF(t)\Big|_a^b - \int_a^b F(t)dt = b - \int_a^b F(t)dt$$

$$E[X] = E[Y] \Leftrightarrow \int_a^b F(t)dt = \int_a^b G(t)dt$$

Break up the integrals: $\int_a^T F(t)dt + \int_T^b F(t)dt = \int_a^T G(t)dt + \int_T^b G(t)dt \quad \forall \; T \in [a,b]$

$$\therefore \int_a^T F(t)dt \leq \int_a^T G(t)dt \Leftrightarrow \int_T^b F(t)dt \geq \int_T^b G(t)dt \quad \forall \; T \in [a,b]$$

English - this is all a fancy way to describe a see-saw (fixed support), in order for the kids to play on it, they have to weigh about the same (mean at pivot point); that means if you replace one kid with a fat kid, you have to replace the other kid with a fat kid (the kids are the "weight" in the tails)

CP 6.2 (Rothchild-Stiglitz) - [7] $\Leftrightarrow$ [5] $\Leftrightarrow$ [2]

Proof:

(a) [5] $\Rightarrow$ [2]

Assume [5]: $y \overset{d}{=} x + \varepsilon$ and $E[\varepsilon \mid x] = 0 \;\; \forall \; x$

Assume $u(\cdot)$ is an increasing, concave function

A little statistics trick: $E[u(y)] = E\{E[u(y) \mid x]\}$

Sub $y \overset{d}{=} x + \varepsilon$: $E[u(y)] = E\{E[u(x+\varepsilon) \mid x]\}$

Use fact that $u(\cdot)$ is an increasing, concave function: $E\{E[u(x+\varepsilon) \mid x]\} \le E\{E[u(x) \mid x]\}$

Stat trick again: $E\{E[u(x) \mid x]\} = E[u(x)]$

$\therefore$ we have $E[x] = E[y]$

(b) [2] $\Rightarrow$ [7]... proof uses more fancy calculus developed in 6.1 and previous pages (see class notes 6.4 if you really care)

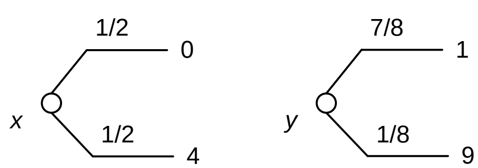
Point - go back to example on p.8; want to say one activity is more risky than another; if they have the same expected value, the three statements ([2], [5], [7]) try to define riskiness:

[2] - any "greedy" (increasing), risk averse (concave) person will prefer less risky option

[5] - more risky activity can be modeled as the less risky activity plus a noise term

[7] - more risky activity is a mean preserving spread of the less risky activity

Note: later we'll use mean preserving spread of likelihood ratios as key for determining when information source is better (it spreads the posterior probabilities making the information better), but for risk, spreading probability is bad



Suppose $u(z) = \sqrt{z}$ (this is increasing and concave)... as shown on p.8, that means $y \succ x$

Now pick a different increasing and concave function:

$$u(z) = \begin{cases} z & \text{if } z \le 8 \\ 8 + 0.1(z-8) & \text{otherwise} \end{cases}$$

$E[u(x)] = \frac{1}{2}(0) + \frac{1}{2}(4) = 2$

$E[u(y)] = \frac{7}{8}(1) + \frac{1}{8}(8 + 0.1) < 2$... so now $x \succ y$

These gambles are too different to compare (neither is a mean preserving spread of the other)

CP 6.3 - Let X be the set of all random variables with equal means and support confined to $[a,b]$. There does not exist a function $H : X \to R$ (real numbers) such that

$H(x) > H(y) \Leftrightarrow x$ is strictly more risky than y for all $x, y \in X$

English - can't define risk

Solution - usually restrict preferences to include mean and variance or restrict the set of gambles so we can more easily determine risk

Stochastic Dominance - another way to try to define "riskiness"

Background - $y = x + \varepsilon$, $\varepsilon \le 0$... this implies $E[u(x+w)] \ge E[u(y+w)]$ $\forall$ random variables w given $u(\cdot)$ is increasing and concave

First Order Stochastic Dominance (FOSD) - equivalent definitions:

(a) $y \overset{d}{=} x + \varepsilon$, $\varepsilon \le 0$
(b) $E[u(x)] \ge E[u(y)]$ $\forall$ increasing u
(c) $F(t) \le G(t)$ $\forall$ $t \in [a,b]$
(where $F(t)$ and $G(t)$ are CDFs of x and y, respectively)

Second Order Stochastic Dominance (SOSD) - equivalent definitions:

[a] $y \overset{d}{=} x + \varepsilon + \delta$, $\varepsilon \le 0$, $E[\delta \mid x + \varepsilon] = 0$
[b] $E[u(x)] \ge E[u(y)]$ $\forall$ increasing, concave u
[c] $\int_a^T [F(t) - G(t)] dt \le 0$ $\forall$

Proofs of the equivalence are sketched in class notes (pp.6.6 and 6.7) and detailed in Ingersoll's appendix

Example - never really got any!

Summary
x dominates y $\Rightarrow$ x FOSD y $\Rightarrow$ x SOSD y
y as risky as (or riskier than) x $\Rightarrow$ x SOSD y

"Risk is like garbage. Think of it generically like a commodity. No one every says one bundle is always better than another for every set of preferences. It's the same with risk."