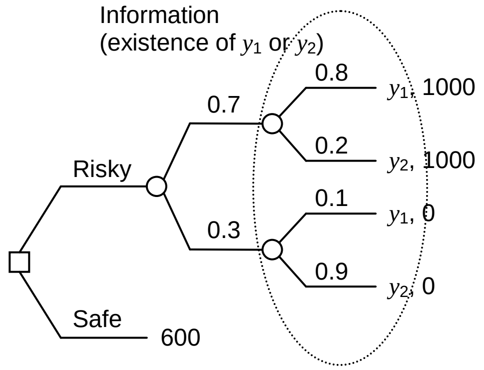
Theory of Choice

Background

Decision Theory



$$\Pr[1000] = 0.7$$

$$\Pr[1000 \mid y_1] = \frac{.56}{.56+.03} = \frac{56}{59} \quad \textbf{(Bayes' Rule)}$$

$$\Pr[1000 \mid y_2] = \frac{.14}{.14+.27} = \frac{14}{41} \quad \text{(Bayes' Rule)}$$

$$\Pr[1000 \mid y] = \sum \Pr[1000 \mid y_i] \cdot \Pr[y_i] =$$

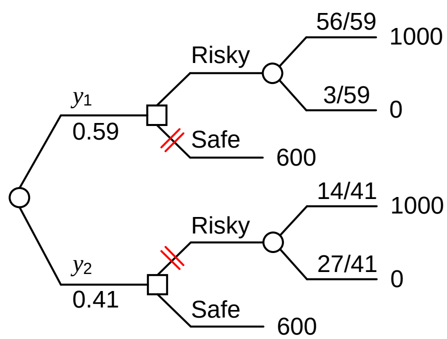
$$\tfrac{.56}{.59}.59 + \tfrac{.14}{.41}.41 = .56 + .14 = .7$$

(**Montingale Property**)

Value of Information - put chance node for information first; requires use of Bayes' Rule to figure out prob of each information event (y_1 and y_2) and to determine conditional probabilities of payoffs given these events (shown above)

For risk neutral (constant absolute risk aversion), value of info is change in certainty equivalent; for other cases, have to figure payoffs with deduction for cost of info and set it equal to the expected utility without information

Risk Neutrality - $u(x) = x$; separates payoffs and probabilities; allows us to use expected payoffs (rather than expected utilities)



$$E[x \mid \text{risky}, y_1] = \tfrac{56}{59}1000 = 949.2 > 600$$

$$E[x \mid \text{risky}, y_2] = \tfrac{14}{41}1000 = 341.5 < 600$$

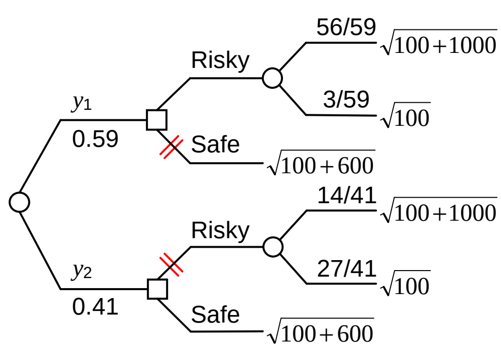
$$E[x \mid \text{info}] = .59(949.2) + .41(600) = 806$$

$E[x] = 700$ (no info)

Value of info (with risk neutrality):
806 - 700 = <u>106</u>

Risk Aversion - $u(x) = \sqrt{100 + x}$; utility of payoff is convex function; **Note:** square root function has decreasing risk aversion (as $x\uparrow$, concavity$\downarrow$... becomes "affine" (linear))

Certainty Equivalent - certain payoff which is indifferent to lottery



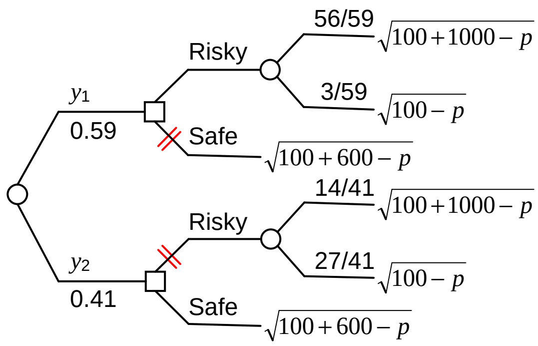
$$E[u \mid \text{risky}] = .7\sqrt{100+1000} + .3\sqrt{100} = 26.216 \approx \sqrt{687}$$

$= \sqrt{100+587}$ $\therefore$ Certainty Equivalent is 587

$$E[u \mid \text{safe}] = \sqrt{100+600} = \sqrt{700}$$

$$E[u \mid \text{info}] = .59(\tfrac{56}{59}\sqrt{1100} + \tfrac{3}{59}\sqrt{100}) + .41\sqrt{700} \approx$$

$\sqrt{865} = \sqrt{100+765}$ $\therefore$ C.E. is 765



$0.56\sqrt{1100-p}+0.03\sqrt{100-p}+0.41\sqrt{700-p}=$
$\sqrt{865}$

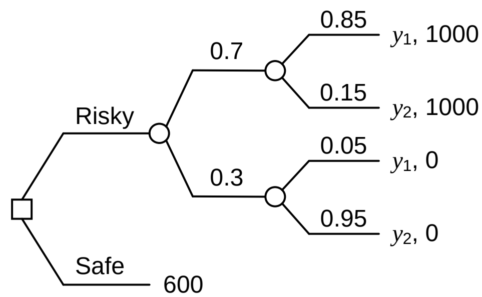
Solve numerically: p = 17.36

Value of info (with this form of risk aversion) is
<u>17.36</u>

Risk Premium - expected value of lottery minus certainty equivalent; 700 - 587 = 112

Really Risk Averse - will never act on the information because will also choose safe, ∴ value of information will be zero (doesn't change the decision so it doesn't change the expected payoff)

Blackwell Theorem - can't compare all information, just information on same problem; for example, the information shown on the right is better than the information in the previous problem because it is more accurate in predicting y_1 and y_2



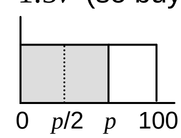
Bilateral Trade -

No Trade Example - value to seller is v ~ U(0,100); value to buyer is $\hat{v}=1.5v$ (so buyer always values it more than seller and trade should take place

Assume buyer offers p (take it or leave it offer)

Seller accepts *iff* $p \geq v$

But $E[v \mid p \geq v] = p/2$ (see picture)



$\therefore\ 1.5E[v \mid p \geq v] - p = \frac{3}{2}\frac{p}{2} - p = -\frac{1}{4}p$ ... ∴ buyer won't buy

We'll study this more in LM Chpt 2

Trade with Second Best Efficiency -

		x_1	x_2	(outcome)
Service	H	.2	.8	
(effort)	L	1	0	

Buyer is risk neutral

Seller is risk averse: $u(\$,a)=\sqrt{\$}-v(a)$, where $v(H)=10$ and $v(L)=0$

Opportunity Cost: $\bar{u}=15$, utility of no trade; must get at least this much to sell

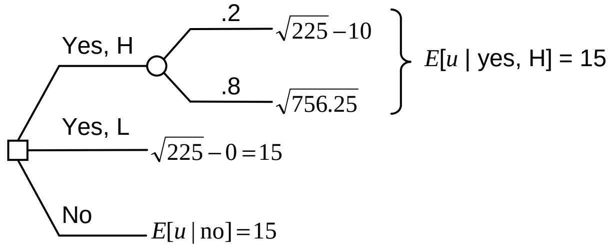
Contract on <u>Service</u> - assumes you see the service

H - if buyer wants H, he has to pay \$ such that $\sqrt{\$}-10=15 \Rightarrow \sqrt{\$}=25 \Rightarrow$
$\$=25^2=625$

L - if buyer wants L, he has to pay \$ such that $\sqrt{\$}=15 \Rightarrow \$=15^2=225$

Contract on <u>Outcome</u> - if buyer can't observe effort, he has to base compensation on output ("a noisy indicator of effort"); needs to ensure incentive compatibility

Incentive Compatibility - look at decision tree for worker; set up payoffs so that worker is indifferent between his choices (or just slightly better for the choice you want him to pick):



Compensating Wage Differential - to get $E[u$ | yes, H] = 15, buyer has to use payments indicated above; that means expected incentive payment for high quality service $E[I \mid H]$ = .2(225) + .8(756.25) = 650... **note:** this is more than the payment for high service when contracting on service (625); the difference is the compensating wage differential

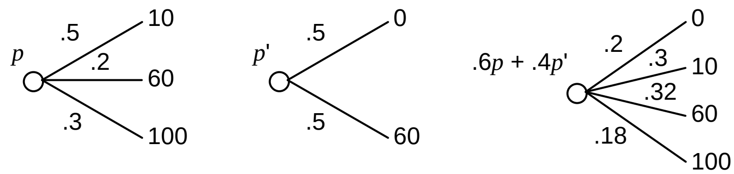
We'll study this more in LM Chpt 4

Preference Relations (Kreps Chpts 1 & 2)

Single-Person Choice Theory - also called **Decision Theory** or **Preference Theory**

Simple Probability Distribution - finite number of outcomes, each associated with a probability; sum of probabilities equals 1

Mixture of Distributions - if p & p' are probability distributions and $a \in (0,1)$, then $ap + (1 - a)p'$ is a new probability distribution, the "a, 1 - a mixture of p and p'"



Totrep - Trade-Off Talking Rational Economic Person

General Method -

1. **Choice set** X is identified
2. **Axioms** concerning Totrep's preferences among members of X are proposed
3. **Representation** Theorem is stated; usually a function from choice set to real numbers (called a **utility function**)
   a. Axioms **Sufficient** - if axioms hold, then representation is possible
   b. Axioms **Necessary** - if representation holds, then axioms must hold
4. **Uniqueness Result** - characterizes extent to which two similarly structured representations of a given preference relation can vary

Axioms Should Be

Consistent - set of axioms that can be satisfied simultaneously

Independent - set of axioms where no subset of them implies the others (i.e., no overlap)

Intuitive - easy to understand

Example - **Strict Preference**; binary relation $\succ$; $p \succ p'$ means Totrep strictly prefers p to p'

<u>Axiom 1.1</u> - if $p \succ p'$, then not $p' \succ p$ (**Asymmetry**)

<u>Axiom 1.2</u> - if not $p \succ p'$ and not $p' \succ p''$, then not $p \succ p''$ (**Negative Transitivity**)

<u>Axiom 1.3</u> - if δ_r denotes lottery which gives prize r with certainty, then $r > r' \Rightarrow \delta_r \succ \delta_{r'}$

<u>Axiom 1.4</u> - if $p \succ p' \succ p''$, then there exist a and b in (0,1) such that
$ap + (1 - a)p'' \succ p' \succ bp + (1 - b)p''$

<u>Axiom 1.5</u> - if $p \succ p'$, then for all a $\in$ (0,1) and all p''
$ap + (1 - a)p'' \succ ap' + (1 - a)p''$

<u>Theorem 1.6</u> (representation) - binary relation $\succ$ on P satisfies Axioms 1.1 to 1.5 if an only if there exists a strictly increasing function u: [0,100] $\rightarrow R$ such that
$p \succ p' \text{ iff } E_p[u] > E_{p'}[u]$

<u>Theorem 1.8</u> (uniqueness) - suppose $\succ$ on P satisfies Axioms 1.1 to 1.5 and u and u' are two representations, then there exist real numbers $c > 0$ and d such that $cu + d = u'$ (i.e., "the representation is unique up to a positive affine transformation")

Normative Applications - if axioms hold (i.e., seem reasonable guides to behavior), we can assess utilities using simple lotteries, then combine these simple lotteries to make a choice on more complicated lotteries

Descriptive Applications - if preferences (as revealed by choices) conform to the five axioms, we can model behavior <u>as if</u> people are expected utility maximizes

Definitions

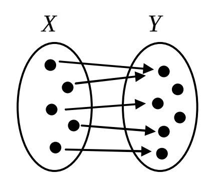
Cartesian Product - $X \times X$; all ordered pairs (x, y), where both x and y are from X

Binary Relation - $B \subseteq X \times X$ binary relation on X ; <u>English</u>: a binary relation says how any two things from the set X are related

Notation - $(x, y) \in B$ or $x\,B\,y$; if $(x, y) \notin B$, then we write not $x\,B\,y$ or $x\,\tilde{B}\,y$

Example - $X = \{1,2,3\}$; binary relation P (or $\succ$, strict preference) is $\{(3,2),(3,1),(2,1)\}$

Function - $f \subseteq X \times Y$ is a function from X into Y ($f: X \rightarrow Y$) if



(1) $\exists\, y \in Y \ni$ (such that) $x\,f\,y$ (i.e., $f(x) = y$) $\forall\, x \in X$; <u>English</u>: every x maps to a y (i.e., in picture, an arrow leaves from each x)

(2) $x\,f\,y$ and $x\,f\,z \Rightarrow y = z$; <u>English</u>: each x maps to a single y (can have several x's map to the same y, but not the other way: 2 y's mapping to one x is not allowed... in picture, each x has a single arrow)

Representation - $\langle x,B \rangle$ is represented by $f: X \rightarrow Y$ in $\langle y,S \rangle$ if
$x\,B\,y \Leftrightarrow f(x)\,S\,f(y)\ \forall\, x,y \in X$; $S \subseteq Y \times Y$

Homomorphic - $\langle x,B \rangle$ is homomorphic to $\langle y,S \rangle$

Measured - if Y is the real line (R), then we say $\langle x,B \rangle$ is measured by f

Caution - we're trying to use real numbers to represent preferences (which form basis of economic theory); problem is that properties of real numbers may nor may not be properties of the preferences we're trying to represent... we have to be very careful in defining what is a valid representation of preferences (so this will hurt)

Properties - may or may not hold for a given binary relation

Property	**Definition**	**Example**
Reflexive	$x\,B\,x \quad \forall x \in X$	$\geq$
Irreflexive	not $x\,B\,x \quad \forall x \in X$	$>$
Symmetric	$x\,B\,y \Rightarrow y\,B\,x \quad \forall x, y \in X$	$=$

Asymmetric .... $x\,B\,y \Rightarrow \text{not } y\,B\,x \quad \forall x, y \in X$ $>$
Antisymmetric .... $x\,B\,y$ and $y\,B\,x \Rightarrow y = x \quad \forall x, y \in X$ $\geq$
Transitive .... $x\,B\,y$ and $y\,B\,z \Rightarrow x\,B\,z \quad \forall x, y, z \in X$ $>$
Negatively Transitive....... $\text{not } x\,B\,y$ and $\text{not } y\,B\,z \Rightarrow \text{not } x\,B\,z \quad \forall x, y, z \in X$ $>$
Using contrapositive: $x\,B\,z \Rightarrow x\,B\,y$ or $y\,B\,z \quad \forall x, y \in X$
Complete (Connected) .... $x\,B\,y$ or $y\,B\,x$ or both $\forall x, y \in X$ ("or" not exclusive)
Another way: $\text{not } x\,B\,y \Rightarrow y\,B\,x \quad \forall x, y \in X$
Weakly Connected .... $x = y$ or $x\,B\,y$ or $y\,B\,x \quad \forall x, y \in X$
Acyclic............................ $x_1\,B\,x_2,\ x_2\,B\,x_3,\ \ldots,\ x_{n-1}\,B\,x_n \Rightarrow x_1 \neq x_n$

Contrapositive - ~B $\Rightarrow$ ~A; logically equivalent to A $\Rightarrow$ B; used for alternate definition of negatively transitive which is easier to use sometimes (proposition 2.1)

Preference Relation - definition 2.2: binary relation $\succ$ on X is a preference relation if it is asymmetric and negatively transitive

Proposition 2.3 - if $\succ$ is a preference relation, then $\succ$ is irreflexive, transitive, and acyclic
Proof: (a) asymmetry directly implies irreflexivity
(b) suppose $x \succ y$ and $y \succ z$;
by negative transitivity (alternate definition) $x \succ y \Rightarrow x \succ z$ or $z \succ y$
but $z \succ y$ violates asymmetry because we assumed $y \succ z$
$\therefore$ must have $x \succ z$ which gives us transitivity
(c) If $x_1 \succ x_2,\ x_2 \succ x_3,\ \ldots,\ x_{n-1} \succ x_n$ then $x_1 \succ x_n$ by transitivity
From asymmetry: not $x_n \succ x_1$
From irreflexivity: $x_1 \neq x_n$ which means we have acyclicity

Example - negatively transitive alone does not imply transitive
X = {1,2,3}
B = {(1,2), (2,3), (3,1), (3,2), (2,1)}
This is clearly not transitive because 1 B 2 and 2 B 3, but 3 B 1 (rather than 1 B 3)
Check for negatively transitive using alternate definition... **bold** terms are in B
$x\,B\,z \Rightarrow x\,B\,y$ or $y\,B\,z$
(1,2) $\Rightarrow$ (1,3) or **(3,2)**
(2,3) $\Rightarrow$ **(2,1)** or (1,3)
(3,1) $\Rightarrow$ **(3,2)** or **(2,1)**
(3,2) $\Rightarrow$ **(3,1)** or **(1,2)**
(2,1) $\Rightarrow$ **(2,3)** or **(3,1)**
B is negatively transitive since condition holds for all elements of B

Weak Preference - $x \succeq y$ if $y \not\succ x$
Indifference - $x \sim y$ if $x \not\succ y$ and $y \not\succ x$

Proposition 2.4 - if $\succ$ is a preference relation, then
(a) $\forall$ x and y, exactly one of $x \succ y$, $y \succ x$ or $x \sim y$ holds (complete)
(b) $\succeq$ is complete and transitive
(c) ~ is reflexive, symmetric, and transitive
(d) $w \succ x$, $x \sim y$, and $y \succ z$ (can be written $w \succ x \sim y \succ z$) $\Rightarrow w \succ y$ and $x \succ z$

(e) $x \succeq y \Leftrightarrow x \succ y$ or $x \sim y$
(f) $x \succeq y$ and $y \succeq x \Rightarrow x \sim y$

Subsets - (not in book); if $\succ \subseteq X \times X$ is a preference relation and $\hat{X} \subseteq X$, then $\hat{\succ} \equiv \succ \cap \hat{X} \times \hat{X}$ is a preference relation (prove in HW)

Example (from notes)

X = {1,2,3,4}

$\succ$ = {(4,3), (4,2), (4,1), (3,2), (3,1), (2,1)}

$\hat{X}$ = {1,2,3}

$\succ \cap \hat{X} \times \hat{X}$ = {(3,2), (3,1), (2,1)}

Revealed Preference - individual's choice behavior reveals his preferences

Choice Function - if Totrep is offered his choice of anything in the set A, he says that any member of $c(A)$ will do; formally, choice function is function $c : X \to X$ such that for all $A \subseteq X$, $c(A) \subseteq A$; easier definition: $c(A, \succ) \equiv \{x \in A : \forall y \in A, y \not\succ x\}$

Proposition 2.8 - for a binary relation $\succ$, $c(\cdot, \succ)$ is a choice function *iff* $\succ$ is acyclic

Houthakker's Axiom - if x and y are both in A and B and if $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$; i.e., if x is sometimes chosen when y is available, then whenever y is chosen and x is available, x is also chosen; can be broken into 2 parts:

Sen's Property α - if $x \in B \subseteq A$ and $x \in c(A)$, then $x \in c(B)$

Example - if the world champion in some game is a Pakistani, then he must also be the champion of Pakistan

Sen's Property β - if $x, y \in c(A)$, $A \subseteq B$ and $y \in c(B)$, then $x \in c(B)$

Example - if Bo and Luke are co-champions of Hazard County and Bo is champion at the state level, the Luke is also a champion at the state level

Proposition 2.13 & 14 (combined) - $\succ$ is a preference relation $\Leftrightarrow$ $c(\cdot, \succ)$ satisfies Houthakker's axiom

Ordinal Utility (Kreps Chpt 3)

Purpose - looking for a numerical representation of binary relation $\succ$

Proposition 3.2 (Cantor Theorem) - if X is a finite set, $\succ$ is a preference relation *iff* $\exists\, u : X \to R$ such that $x \succ y \Leftrightarrow u(x) > u(y)$ $\forall\, x, y \in X$ (this is referred to as "the property" below)

<u>Proof</u>: first look at <u>necessity</u> (i.e., show if function exists with the property above, then preference relation exists): assume existence of function u such that $x \succ y \Leftrightarrow u(x) > u(y)$

$x \succ y \Leftrightarrow u(x) > u(y) \Rightarrow \text{not } \{u(y) > u(x)\} \Rightarrow \text{not } y \succ x$... $\therefore \succ$ is asymmetric

$\text{not } x \succ y$ and $\text{not } y \succ z \Rightarrow \text{not } \{u(x) > u(y) > u(z)\}$... which is same as $u(x) \le u(y) \le u(z)$

Based on transitivity of numbers then, $u(x) \le u(z)$

That's the same as $\text{not } \{u(x) > u(z)\} \Rightarrow \text{not } x \succ z$... $\therefore \succ$ is negatively transitive

Since $\succ$ is asymmetric and negatively transitive, it is a preference relation

Now look at <u>sufficiency</u> (i.e., show if $\succ$ is preference relation there exists a function u with the property above)... using proof by induction

Intuition - given representation u' for preference relation $\succ$ on set X', determine where x^0 would fall in X' and then develop a relation between u and u' that allows u to satisfy the property above for $X = X' + x^0$

Assume X has n elements (n = 1,2,3,...) and $\succ$ is a preference relation on X

Trick... it's actually easier to prove if we set the range of u to the interval (0,1), which is allowed because u is unique up to a strictly increasing transformation

Look at $n = 1$... that means X has a single element; for $x, y \in X$, $x \succ y$ is not possible (by asymmetry) and $u(x) > u(y)$ is not possible (by definition of a function) so the property holds trivially (vacuously true)

Now pick any $n > 1$ and assume result is true for sets of size $n - 1$ (induction hypothesis)

For set X with n elements, pick any arbitrary element x^0 and let $X' \equiv X \setminus x^0$ (i.e, the set X without the element x^0)

X' has $n - 1$ elements so by induction hypothesis $\exists\, u' : X' \to (0,1)$ such that the property above holds for all $x, y \in X'$

To look at adding x^0 to X', there are four cases... must check that property holds:

1. Indifferent: $\exists\, x'' \in X'$ such that $x^0 \sim x''$

   Define $u(x) = \begin{cases} u'(x) & \text{if } x \in X' \\ u'(x'') & \text{if } x = x^0 \end{cases}$

   If $x, y \in X'$ then $x \succ y \Leftrightarrow u'(x) > u'(y)$ (by induction hypothesis) $\Leftrightarrow u(x) > u(y)$ (because $u \equiv u'$ on X')

   If $x = y = x^0$, the both $x \succ y$ and $u(x) > u(y)$ are impossible so the property is vacuously true

   If $x \in X'$ and $y = x^0$, then $x \succ y \Leftrightarrow x \succ x''$ (since $x'' \sim y = x^0$... see part d of Prop 2.4) $x \succ x'' \Leftrightarrow u'(x) > u'(x'') \Leftrightarrow u(x) > u(x^0)$ (since $u(x) = u'(x)$ and $u(x^0) = u'(x'')$)

   If $x = x^0$ and $y \in X'$, then $x \succ y \Leftrightarrow x'' \succ y$ (since $x'' \sim x = x^0$... see part d of Prop 2.4) $x'' \succ y \Leftrightarrow u'(x'') > u'(y) \Leftrightarrow u(x^0) > u(y)$ (since $u(x^0) = u'(x'')$ and $u(y) = u'(y)$)

2. Best: $x^0 \succ x\ \forall\, x \in X'$

   Define $u(x) = \begin{cases} u'(x) & \text{if } x \in X' \\ \frac{1}{2}\left[\max\limits_{x \in X'} u'(x) + 1\right] & \text{if } x = x^0 \end{cases}$

   (Note: $u'(x) < 1\ \forall\, x \in X'$ so the average of the largest value of u and 1 will greater than the largest value of u)

   If $x, y \in X'$ then same as case 1

   If $x = y = x^0$, then same as case 1

   If $x \in X'$ and $y = x^0$, then $x \succ y$ is impossible by construction; also can't have $u(x) > u(y)$ for same reason

   If $x = x^0$ and $y \in X'$, then $x \succ y$ (by hypothesis of this case) and $u(x) > u(y)$ by construction

3. Worst: $x \succ x^0\ \forall\, x \in X'$

   Define $u(x) = \begin{cases} u'(x) & \text{if } x \in X' \\ \frac{1}{2}\left[\min\limits_{x \in X'} u'(x)\right] & \text{if } x = x^0 \end{cases}$

   (Note: taking 1/2 of min $u(x)$ guarantees $u(x^0)$ will be less than $u(x)\ \forall\, x \in X'$)

   Given the specification of $u(x)$ above, the rest is the same as case 2 above.

4. In Between: $x \not\sim x^0\ \forall\, x \in X'$ and $\exists\, x \in X'$ such that $x \succ x^0$ and $\exists\, y \in X'$ such that $x^0 \succ y$

   Define $\bar{x}$ such that $u'(\bar{x}) = \min\limits_{x \in X' : x \succ x^0} u'(x)$ (i.e., "worst" thing in X' that's better than x^0)

   Define $\underline{x}$ such that $u'(\underline{x}) = \max\limits_{x \in X' : x^0 \succ x} u'(x)$ (i.e., "best" thing in X' that's worse than x^0)

   Define $\begin{cases} u'(x) & \text{if } x \in X' \\ & \end{cases}$

$u(x) = \frac{1}{2}\left[u'(\bar{x}) + u'(\underline{x})\right]$ if $x = x^0$

Note: if $x \in X'$ is such that $x \succ x^0$, then $u'(x) \geq u'(\bar{x})$ so that $x \succeq \bar{x}$; similar logic has $x^0 \succ x$ imply $\underline{x} \succeq x$; since $\bar{x} \succ x^0 \succ \underline{x}$, transitivity says $\bar{x} \succ \underline{x}$ so $u(\bar{x}) > u(\underline{x})$ $\therefore$ $u(\bar{x}) > \frac{1}{2}\left[u(\bar{x}) + u(\underline{x})\right] = u(x^0) > u(\underline{x})$

If $x, y \in X'$ then same as case 1

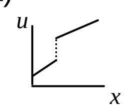
If $x = y = x^0$, then same as case 1

If $x = x^0$ and $y \in X'$, then $x \succ y \Leftrightarrow x^0 = x \succ \underline{x} \succeq y \Leftrightarrow u(x^0) = u(x) > u(\underline{x}) \geq u(y)$

If $x \in X'$ and $y = x^0$, then $x \succ y \Leftrightarrow x \succeq \bar{x} \succ y = x^0 \Leftrightarrow u(x) \geq u(\bar{x}) > u(y) = u(x^0)$

Examples - (from lesson 2 notes; not in book)

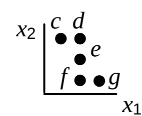
1. $X = [0, 1]$; $\succ \equiv >$... more is better

   Could use $u(x) = x$ or $u(x) = \sqrt{x}$ or 

2. $X = [0, 1] \times [0, 1]$, $x \succ y \Leftrightarrow x_1 + x_2 > y_1 + y_2$

   Could use $u(\mathbf{x}) = x_1 + x_2$ or $u(\mathbf{x}) = \sqrt{x_1 + x_2}$ or $u(\mathbf{x}) = \alpha + \beta(x_1 + x_2)$ ($\beta > 0$),

   but not $u(\mathbf{x}) = \sqrt{x_1} + \sqrt{x_2}$ (doesn't preserve order)

3. $X = [0, 1] \times [0, 1]$, $x \succ y \Leftrightarrow x_1 > y_1$ or ($x_1 = y_1$ and $x_2 > y_2$) (**Lexicographic**) 

   Suppose a measure exists, $\phi(x_1, x_2)$

   Formally - $\phi(x,1) > \phi(x,0)$ $\forall\, x \in [0,1]$ (e.g., points d and f in picture)

   For each x, we have a rational $r(x)$ such that $\phi(x,1) > r(x) > \phi(x,0)$

   That is, uncountably infinite number of points between uncountably infinite number of points

   Intuition - from picture $g \succ d \succ e \succ f \succ c$

   c and g are arbitrarily close in x_1, but have uncountably infinite number of utilities in between them while maintaining $u(g) > u(c)$

   Problem - this is a valid preference relation, but Cantor Theorem (Prop 3.2) doesn't help... so we need to strengthen it

Proposition 3.3 - if X is a countably infinite set, $\succ$ is a preference relation *iff* $\exists\, u : X \to R$ s.t. $x \succ y \Leftrightarrow u(x) > u(y)$ $\forall\, x, y \in X$

Proposition 3.4 - for arbitrary set X, $\succ$ is a preference relation *iff* $\exists\, u : X \to R$ such that $x \succ y \Leftrightarrow u(x) > u(y)$ $\forall\, x, y \in X$

Order Dense - suppose $\succ$ is a binary relation on a set X; $Z \subseteq X$ is $\succ$-order dense if for all $x, y \in X$ such that $x \succ y$, there exists some $z \in Z$ with $x \succeq z \succeq y$

English: if we take any two elements in X, there's always something in Z that is between them

Proposition 3.5 - for arbitrary set X, $\exists\, u : X \to R$ such that $x \succ y \Leftrightarrow u(x) > u(y)$ $\forall\, x, y \in X$ *iff*

A1. $\succ$ is a preference relation, and

A2. $\exists$ a <u>countable</u> $\succ$-order dense subset of X

Uniqueness - (for all the propositions above) u is unique up to a strictly increasing transformation (which preserves order [>])

Formally - Let $f : u(x) \to R$ be strictly increasing $\therefore\ x \succ y \Leftrightarrow u(x) > u(y) \Leftrightarrow f(u(x)) > f(u(y))\ \forall\ x, y \in X$

Proof: Suppose u and $\hat{u}$ both represent $\succ$;

Define $\bar{f} : u(X) \to R$ via $\bar{f}[u(x)] = \hat{u}(x)$,

But $u(x) > u(y) \Leftrightarrow \hat{u}(x) \Leftrightarrow \hat{u}(y)\ \therefore\ \bar{f}$ is strictly increasing on $u(X)$

Continuity and additivity (not in book)

Purpose - measuring preferences, but representing with utility; it's possible that utility function is continuous (and/or additive) because of properties of real numbers, but we need analogous property in preferences... "we never get anything for free"

More Info - see Debreu (Theory of Value) or Mas-Colell

CP 2.1 - rewrites Prop 3.5 replacing A2 with A2 & A3 and making u continuous

Let $Z \subseteq R$ and $X = Z^n$ (i.e., X is commodity space); if

A1. $\succ \subseteq X$ x X is a preference relation (i.e., compares any two bundles in X)

A2. $\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x} \succ \mathbf{y}$ (i.e., usual assumption that more is better)

A3. $\mathbf{x} \succ \mathbf{y} \succ \mathbf{z} \Rightarrow \exists\ \alpha, \beta \in (0,1)$ such that $\alpha\mathbf{x} + (1-\alpha)\mathbf{z} > \mathbf{y} > \beta\mathbf{x} + (1-\beta)\mathbf{z}$ (i.e., nothing is too bad or too good), then

$\exists$ continuous, strictly increasing $u : X \to R$ such that $\mathbf{x} \succ \mathbf{y} \Leftrightarrow u(\mathbf{x}) > u(\mathbf{y})\ \ \forall\ \mathbf{x}, \mathbf{y} \in X$

Definitions - background for CP 2.3; there's lots 'o math theory behind these things economists tend to take for granted

Topological Space - $\langle X, \zeta \rangle$ is a topological space where ζ is a set of subsets of X such that

$\varnothing \in \zeta$

$X \in \zeta$

ζ closed under arbitrary unions

ζ closed under finite intersection

Continuous Relation - if $\langle X, \zeta \rangle$ is a topological space and $\succ \subseteq X$ x X is a preference relations, then $\succ$ is continuous (on the topology ζ) if $\{y : y \succ x\} \in \zeta$ and $\{y : x \succ y\} \in \zeta\ \forall\ x \in X$

Continuous Function - if $\langle X, \zeta \rangle$ is a topological space, then function $f : X \to R$ is continuous (in the topology ζ) if $A \in U \Rightarrow \{x : x \in X, f(x) \in A\} \in \zeta$

Note: U is set of subsets for real line, R; so topology space for real line is $\langle R, U \rangle$

Implication - preference relation that admits a continuous measure is also continuous... this is built into CP 2.1

Intuitively - if representation is smooth in domain (R), then it's smooth in the domain (X)... so continuous representation (e.g., utility function) means there's continuous preferences

Examples of Additivity - these are based on real numbers, not preference relations

$\Pr[A \cup B] = \Pr[A] + \Pr[B]$ (if A and B are independent)

$u(\$, a) = \sqrt{\$} + u(a)$

$u(\bullet) = \sum (1+r)^{-t} V(X_t)$

$u = E[u \mid \bullet] = \sum u(x) \cdot p(x)$

OLS Regression... e.g., look at productivity of semi trailer given type of trailer (flat bed, box, car carrier, etc.): $prod = a + b \cdot type + c \cdot dummies$ (truck type, truck year, driver experience, etc.)... the additivity here assumes there is no interaction between the type of trailer and the other variables

Empirical Relation System - $\langle x_1, x_2, \ldots, x_n, \succ \rangle$, $n \geq 2$

Natural Concatenation Operator - method for combining measure of two objects (e.g., for weight, we can simply add the two weights or put both people on the scale at the same time)

Additive Conjoint Measure - one method for concatenation operator is to simultaneously measure objects components: $u(x) = \sum_{j=1}^{n} u_j(x_j)$

Independence Idea - preference ordering should be same for beer regardless of amount of cheese (e.g., if 1 Coors with 1 cheese beats 1 Bud with 1 cheese, then 1 Coors with 3 cheese beats 1 Bud with 3 cheese)

Formally - $X = X_1 \times X_2$ and $\succ$ is a preference relation, then define $Q = \succ \cap X_1 \times \{\bar{x}_2\}$, where $\bar{x}_2 \in X_2$ (i.e., hold second commodity fixed);an example of independence is $u(\mathbf{x}) = f(x_1) + g(x_2)$; in this case $(x_1, \bar{x}_2) \succ (x_1', \bar{x}_2)$ becomes $f(x_1) + g(\bar{x}_2) > f(x_1') + g(\bar{x}_2)$ which simplifies to $f(x_1) > f(x_1')$ $\therefore$ it doesn't matter what value we use for x_2 (as long as it's the same in both bundles)

Expand - just showed independence where 1 commodity (of 2) was held constant; can expand to any "reasonable" m commodities (of n):

Preference relation $\succ$ on $X_1 \times X_2 \times \ldots \times X_n$ is independent if for every $m \subseteq \{1, 2, \ldots, n\}$ the ordering $\succ_m$ induced by $\succ$ on $\times_{i \in m} X_i$ for fixed elements $\bar{x}_j \in X_j \;\; \forall j \notin m$ is unaffected by the choice of fixed elements

Essential - component $j \in m = \{1, 2, \ldots, n\}$ is essential if $\exists\, x_k \in X_k$, $k \in \{1, 2, \ldots, n\} \setminus \{j\}$ and $x_j', x_j'' \in X_j$ such that $(x_1, \ldots, x_{j-1}, x_j', x_{j+1}, \ldots, x_n) \sim (x_1, \ldots, x_{j-1}, x_j'', x_{j+1}, \ldots, x_n)$ is false

English: if we change the amount of good j with all else equal and the bundles are indifferent then j^{th} good is not essential (i.e., we care about amount of good j)

CP 2.2 - Given preference relation $\succ$ on $X_1 \times X_2 \times \ldots \times X_n$ is independent, component j is essential *iff* $\exists\, x_j', x_j'' \in X_j$ such that $x_j' \succ x_j''$ (prove in HW)

Connected - a set is connected if it cannot be partitioned into two nonempty subsets such that each subset has no points in common with the closure of the other

Formally - (casual formal) let $\langle X_i, \zeta_i \rangle$ be a topological space, $i = 1, \ldots, n$; $\langle X, \zeta \rangle$ is a topological space and ζ is the **product topology** when

(1) $a_i \in \zeta_i$, I = 1, ..., $n \Rightarrow \prod_{i=1}^{n} a_i \in \zeta$

(2) ζ is closed under arbitrary unions

(3) ??? (not finished in notes)

Purpose - combining topology spaces (like a Cartesian Product) to show total set of possible combinations of choosing from X each period for n periods... grows exponentially

CP 2.3 - Given commodity space $X = X_1$ x X_2 x ... x X_n with each X_i connected and preference relation $\succ \subseteq X$ x X that is independent, continuous and has at least 3 essential components, then

(1) $\exists\, u_i : X_i \rightarrow R$ such that $\mathbf{x} \succ \mathbf{y} \Leftrightarrow \sum u_i(x_i) > \sum u_i(y_i)\ \forall\, \mathbf{x}, \mathbf{y} \in X$

(2) if set of functions $\{\bar{u}_i\}$ also represents $\succ$ in this fashion, $\exists\, \alpha > 0$ and β_i such that

$\bar{u}_i = \alpha u_i + \beta_i$ (i.e., unique up to strictly increasing transformation)

Note: all representations use same multiple to keep tradeoff on margin the same

Homogeneous Product Sets - $X = Z^n$ (e.g., consumption each period)

Stationary - $\succ$ on Z^n is stationary if $\exists\, x \in Z$ such that $\forall\, y_j, z_j \in Z$, j = 1, ..., n-1

$(y_1, ..., y_{n-1}, x) \succ (z_1, ..., z_{n-1}, x) \Leftrightarrow (x, y_1, ..., y_{n-1}) \succ (x, z_1, ..., z_{n-1})$

CP 2.4 - append following to CP 2.3: $X = Z^n$ and $\succ$ stationary, then $u_j = \boldsymbol{\lambda}^{j-1} u$, where λ is unique and u is unique to positive affine transformation

Example - $\lambda = (1 + r)^{-1}$, then $\lambda^{j-1} = (1 + r)^{1-j}$... i.e., discounted cash flow (or utility)

$$\frac{u_1}{1+r} + \frac{u_2}{(1+r)^2} + \cdots$$