Neoclassical Growth Models

**Model** - mathematical representation of some aspect of the economy; best models are often very simple but convey enormous insight into how the world works

"All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive." - Robert Solow, 1956

**Solow Model**

**Assumptions**
1. Single, **homogeneous good** (output) - this also implies that there is no international trade (have to have at least two different goods for any trade to take place)
2. **Technology is exogenous** - technology available to firms is unaffected by the actions of the firms, including R&D (we'll relax this later); implication is that production function is not shifting
3. **Keynesian Assumption** - individuals save a constant fraction of their income (i.e., people consume in proportion to income)... this assumption is consistent with the data... let the **savings rate** be **s**... total savings is \( sY \)
4. **Labor Force** - population is same as labor force (good enough to have constant labor force participation rate); population grows at constant rate \( n \) \( L(t) = L_0 e^{nt} \)
5. **Perfect Competition** - zero economic profit; price takers in labor market \( (w) \) and capital market \( (r) \)
6. **Constant Returns to Scale** - can use any constant returns production function (i.e., double input produces double output); we'll focus on Cobb-Douglas
   - **Two Inputs** - capital and labor; capital is accumulated endogenously through savings
   - **Constant Elasticity of Substitution (CS)** - production function with constant returns to scale; generalization of Cobb-Douglas; \( Y(t) = F(K,L) = (bK^\alpha + cL^\rho)^{\frac{1}{\rho}} \)
   - **Cobb-Douglas Function** - \( Y(t) = F(K,L) = K^\alpha L^{1-\alpha}, \alpha \in (0,1) \)
   - **Test Constant Returns** - \( F(2K,2L) = (2K)^\alpha(2L)^{1-\alpha} = 2K^\alpha L^{1-\alpha} = 2F(K,L) \)... yep
7. **Constant Depreciation Rate** - capital depreciates at a constant rate \( \delta \)

**Basic Model** - built around two equations: production function & capital accumulation equation

**I. Production Function** - describes how capital inputs, \( K(t) \), combine with labor, \( L(t) \), to produce output, \( Y(t) \)
   - **Capital** - look at corn... some you eat (consumption), some you plant to eat more in the future (capital); capital postpones consumption because certain amount in hopes of higher (but uncertain) future consumption... definition of investment
   - **Max Profits** - hire capital and labor at market prices to maximize profits (Note: we can consider our good to be a numeraire \( (P = 1) \) which means \( r \) and \( w \) are in terms of \( Y \))
     - Max \( F(K,L) - rK - wL \)
   - **First Order Conditions** - \[ \frac{d\pi}{dK} = F_K - r = 0 \] ... marginal product of capital = \( r \)
\[
\frac{d\pi}{dL} = F_L - w = 0 \quad \text{... marginal product of labor } = w
\]

**Using Cobb-Douglas**

\[
F_K = \alpha K^{\alpha-1} L^{1-\alpha} = r \Rightarrow \quad r = \frac{\alpha K^{\alpha} L^{1-\alpha}}{K} = \frac{\alpha Y}{K}
\]

\[
F_L = (1 - \alpha) K^{\alpha} L^{(1-\alpha)-1} = w \Rightarrow \quad w = \frac{(1 - \alpha) K^{\alpha} L^{1-\alpha}}{L} = \frac{(1 - \alpha) Y}{L} \quad \text{... we can use this to get the demand for labor}
\]

**Finding Steady State** — in order to find some form of the production function to give us a steady state, we need to identify variables that won't change over time; in this case, we can focus on \(r\) & \(w\) because (1) empirical data says they’re stable over time [fact 5 from Introduction to Growth], (2) based on the way the production function is set up, if either of these variables grows, the other has to decline and if we’re talking growth rates, that means the other variable will eventually reach zero... not realistic

From first order conditions we have \(w = \frac{(1 - \alpha) Y}{L} \quad \therefore \frac{Y}{L}\) is also constant; that's called the **output per worker**: \(y \equiv \frac{Y}{L}\)

We also have \(r = \frac{\alpha Y}{K} \quad \therefore \frac{Y}{K}\) is also constant; to put things in same terms (per worker or per capita), we'll use a trick: \(\frac{Y}{L} = \frac{y}{\frac{Y}{L}}\); since \(y\) is constant that means **capital per worker**: \(k \equiv \frac{K}{L}\) is also constant

**Results for Production Function**

1. **Zero Profit** — results from constant returns assumption;

   \[
   \text{Profit} = Y - rK - wL = Y - \frac{\alpha Y}{K} - \frac{(1 - \alpha) Y}{L} = Y - \alpha Y - (1 - \alpha) Y = 0
   \]

   \(\therefore\) payments to inputs completely exhaust the value of output produced

2. **Constant Labor & Capital** — as fractions of GDP; results from Cobb-Douglas production function: \(\frac{rK}{Y} = \alpha\) and \(\frac{wL}{Y} = 1 - \alpha\); this agrees with empirical data (see fact 5 from Introduction to Growth)

3. **Growth Rates Equal** — take \(\ln\) of both sides of capital per worker: \(k = K/L\)

   \[
   \ln k = \ln K - \ln L
   \]

   Totally differentiate wrt \(t\):
   \[
   \frac{1}{k} \frac{dk}{dt} = \frac{1}{K} \frac{dK}{dt} - \frac{1}{L} \frac{dL}{dt} \quad \text{or} \quad \dot{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}
   \]

   Those are growth rates; \(k\) is a stable variable in steady state so \(\frac{dk}{dt} = 0\); that means \(\dot{K} = \dot{L} \quad \text{... the growth rate of capital equals the growth rate of labor in the steady state}\)
Per Capita Production - now that we have steady state variables we can work with (k and y), we need to convert the production function to incorporate these variables:  
\[ Y = F(K, L) \]  ... because of constant returns to scale, we can write  
\[ y = \frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F(k, l) \]  ... introduce per capita production function \( f: y = f(k) \)

Using Cobb-Douglas - \( y = f(k) = \frac{K^\alpha L^{1-\alpha}}{L} = K^\alpha L^{1-\alpha} = \left(\frac{K}{L}\right)^\alpha = k^\alpha \)

\[ y' = \alpha k^{\alpha -1} > 0 \] so output per worker always increases with \( k \) increases  
\[ y'' = \alpha(\alpha -1)k^{\alpha -2} < 0 \] (because \( \alpha < 1 \)). We have diminishing returns to capital (the added output per worker by increasing capital per worker declines as we add more capital)

II. Capital Accumulation - the change in the capital stock over time is equal to the infusion of new capital from savings (sY) minus the depreciation of capital  
\[ \frac{dK}{dt} = \dot{K} = sY - \delta K \]

Example - economy starts with output of 100 and capital base of 200; the capital depreciate rate is 5 percent and the savings rate is 30 percent  
Savings (gross investment) = sY = 0.3(100) = 30  
Depreciation = \( \delta K = 0.05(200) = 10 \)  
Capital accumulation (net investment) = sY - \( \delta K = 30 - 10 = 20 \)

Per Capita Capital Accumulation - just like we did with the production function, we want to get this equation to incorporate variables that are constant in the steady state (k and y)  
Divide all terms by \( K \):  
\[ \frac{dK}{dt} = \frac{\dot{K}}{K} = s\frac{Y}{K} - \delta\frac{K}{K} = s\frac{Y}{K} - \delta \]

Where did we see that \( \frac{dK}{dt} \) term before? We got it from differentiation  
\[ \ln k = \ln K - \ln L \] with respect to \( t \) when we showed that the growth rates of capital and labor are equal in the steady state (on previous page); the general (non steady state) result was  
\[ \frac{dK}{dt} = \dot{K} = s\frac{Y}{K} - \delta\frac{K}{K} = s\frac{Y}{K} - \delta \]

From assumption 4 substitute the labor growth rate \( n \):  
\[ \frac{dK}{dt} = \dot{K} = s\frac{Y}{K} - \delta \]

Solve that for \( \frac{dK}{dt} \):  
\[ \frac{dK}{dt} = \dot{K} = s\frac{Y}{K} - \delta \]

Substitute that into the first equation we had:  
\[ \frac{dK}{dt} = s\frac{Y}{K} - \delta \]

To get \( y \) into the equation, use the \( L/L \) trick again:  
\[ \frac{dK}{dt} = s\frac{Y}{K} - \delta \]

Solve for capital accumulation per worker:  
\[ \frac{dK}{dt} = s\frac{Y}{K} - \delta \]

Steady State - in steady state, \( k \) must be constant so \( \frac{dk}{dt} = 0 \Rightarrow s\frac{Y}{K} = (n + \delta)k \)
That is, the gross savings (investment in capital) per worker must equal the lost capital per worker (lost through depreciation and the increase in the number of workers)

**Solve for Growth Rates** - use formulas with standard trick of taking \( \ln \) of both sides, then differentiating

\[
y = k^\alpha \Rightarrow \ln y = \alpha \ln k \Rightarrow \dot{y} = \alpha \frac{k}{y} \quad \text{since} \quad \dot{k} = 0, \text{we must have} \quad \dot{y} = 0
\]

\[
Y = K^\alpha L^{1-\alpha} \Rightarrow \ln Y = \alpha \ln K + (1-\alpha) \ln L \Rightarrow \frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}}{L} = \alpha n + (1-\alpha)n = n \quad \text{... so output grows at same rate as labor force}
\]

**Graphically** - the Solow model is pretty easy to solve graphically... just look at where \( sy \) intersects \( (\delta + n)k \)

**Stable Equilibrium** - if we have \( k \) anywhere other than \( k^* \), it'll eventually adjust to be at \( k^* \); for example, start with \( k < k^* \); this means savings outpaces effective depreciation so we're accumulating capital \( (k \uparrow) \); this continues until \( k = k^* \)

**Algebraically** - solving is a little more complicated; start with steady state equation:

\[
sy = (n + \delta)k
\]

Plug in production per worker function:

\[
sk^\alpha = (n + \delta)k
\]

Solve for \( k \):

\[
k^{1-\alpha} = \frac{s}{n + \delta} \Rightarrow k = \left( \frac{s}{n + \delta} \right)^{1-\alpha}
\]

Plug that back into the production per worker function to solve for \( y \):

\[
y^* = (k^*)^\alpha = \left( \frac{s}{n + \delta} \right)^{\frac{\alpha}{1-\alpha}}
\]

**Advantage of Algebra** - can check model empirically: \( \ln y^* = \frac{\alpha}{1-\alpha} \ln s - \frac{\alpha}{1-\alpha} \ln(n + \delta) \); regression should show parameters \( \beta_1 = \beta_2 \); also can solve for \( \alpha \) and check if \( \alpha < 1 \)

**Summary** - started with two equations:

\[
Y = F(K, L) \quad \text{and} \quad \frac{dK}{dt} = sY - (n + \delta)K
\]

Then introduced output per worker and capital per worker:

\[
y = Y / L \quad \text{and} \quad k = K / L
\]

Modified original equations:

\[
y = f(k) \quad \text{and} \quad \frac{dk}{dt} = sy - (n + \delta)k
\]

Steady state has \( \frac{dk}{dt} = 0 \) so we know

\[
sy = (n + \delta)k
\]

Other things we showed

Growth rates of labor, capital, and output are equal: \( \dot{L} = \dot{K} = \dot{Y} = n \)
Comparative Statics

Increase Savings Rate - $s \uparrow \Rightarrow y \uparrow$ and $k \uparrow$

Note: growth rate in long run doesn’t change since growth rate of capital must equal the growth rate of labor (population); the result will be a higher y (GDP per capita) that grows at the same rate as before the increased savings rate

Decrease Population Growth Rate - $n \downarrow \Rightarrow y \uparrow$ and $k \uparrow$

Decrease Depreciation Rate - $\delta \downarrow \Rightarrow y \uparrow$ and $k \uparrow$

Increase Productivity of Capital - $\alpha \uparrow \Rightarrow y \uparrow$ and $k \uparrow$

Transitional Dynamics - how model evolves (between equilibria)

Algebraically - we already showed:

$$\frac{\dot{y}}{y} = \alpha \frac{k}{k}$$

(which we used to find $\dot{y} = 0$ in steady state; we’re not in steady state now so we have to work with the general version; the other equation we need is the capital per worker accumulation equation which we’ll divide by $k$:

$$\frac{\dot{k}}{k} = \frac{sk^n}{k} - (n + \delta) = \frac{s}{k^{1-n}} - (n + \delta)$$

Now we have two differential equations to describe the transitional dynamics; since diffeq isn’t fun, we’ll just look at the transitions geometrically by plotting that second equation to determine what happens to the growth rate of $k$

Geometrically -

Increase Savings Rate - $s \uparrow$ shifts the $sk^{\alpha-1}$ curve; at the current capital per worker, we aren’t in equilibrium so $\dot{k} / k$ increases (i.e., the rate of growth of capital per worker increases); note form the first equation we showed above that this means the output per worker is also increasing; eventually, we’ll settle back down where $sk^{\alpha-1} = n + \delta$ so $\dot{k} / k$ goes back to zero (so does $\dot{y} / y$); the end result is a one time increase in capital per worker ($k$) and output per worker ($y$), but the growth rate of these remains the same at zero

Decrease Population - if $L \downarrow$ (and $n$ remains constant), we have a one time increase in capital per worker ($k$); this shifts us to the right on the graph and there’s a disparity between $(n + \delta)$ and $sk^{\alpha-1}$ which means $\dot{k} / k < 0$ (which means $\dot{y} / y < 0$); it remains so until capital per worker goes back to the original level; the end result is a one time decrease output per worker ($y$)... this is why we care about unemployment
Problems -

- **Model is Population Driven** - growth rates of output ($Y$) and capital ($K$) equal growth rate of labor ($L$)
- **No Growth in Output per Worker** - model says $y$ doesn't grow in steady state, but data for U.S. says otherwise... this is why we introduce technological change

**Solow Model + Technology**

**Technology** - captured in the production function

**Hicks-Neutral** - $Y = AF(K, L) = F(AK, AL)$

**Capital Augmenting** - $Y = F(AK, L)$

**Labor Augmenting** - $Y = F(K, AL)$

Results of technology are most transparent with labor augmenting technology so that's where we'll focus

**Effective Labor** - $AL$; we use $A$ to represent the level of technology; using the labor augmenting production function, if $A$ changes from 1 to 2, that means workers are twice as productive (we effectively get the same work as if we doubled the number of workers)

**Update Model** - basically have four equations that drive the model

1. $Y = F(K, AL) = K^\alpha (AL)^{1-\alpha}$
2. $A(t) = A_0 e^{\alpha t} \Rightarrow \frac{\dot{A}}{A} = g$ (constant rate of technological advancement)... this means that technological progress is exogenous ("mana from heaven"); technology descends upon the economy automatically and regardless of whatever else is going on in the economy
3. $L(t) = L_0 e^{n t} \Rightarrow \frac{\dot{L}}{L} = n$ (constant labor growth rate)... same as before
4. $\dot{K} = sY - \delta K$ ... same capital accumulation equation as before

**What's Constant?** - divide equation 1 by capital to get output per capital:

$$\frac{Y}{K} = \frac{K^\alpha (AL)^{1-\alpha}}{K} = \left(\frac{AL}{K}\right)^{1-\alpha}$$

Since both $A$ and $L$ grow at constant rates ($g$ and $n$), $K$ must also grow at a constant rate or else $Y/K$ would either go to infinity or go to zero (both are unrealistic); if we invert $AL/K$ we get capital per effective worker:

$$\tilde{k} \equiv K / AL = k / A$$

Similarly, we can define output per effective worker: $\tilde{y} \equiv Y / AL = y / A$; now we can rewrite equation 1:

$$\tilde{y} = \frac{Y}{AL} = \frac{K^\alpha (AL)^{1-\alpha}}{AL} = \left(\frac{K}{AL}\right)^{\alpha} \equiv \tilde{k}^\alpha$$

Output per effective worker and capital per effective worker will be constant in steady state

**Growth Rates** - Do the ln-differentiate trick on $\tilde{k}$ and $\tilde{y}$

$$\ln \tilde{k} = \ln k - \ln A \Rightarrow \frac{\dot{k}}{\tilde{k}} = \frac{k}{\tilde{k}} \frac{\dot{A}}{A} \Rightarrow \frac{\dot{k}}{k} = \frac{\dot{A}}{A} = g \therefore \text{capital per worker ($k$) grows at same rate as technological progress}$$
\[
\ln \tilde{k} = \ln K - \ln A - \ln L \quad \Rightarrow \quad \frac{\dot{\tilde{k}}}{\tilde{k}} = 0 = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \quad \Rightarrow \quad \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g + n
\]

\[
\ln \tilde{y} = \ln y - \ln A \quad \Rightarrow \quad \frac{\dot{\tilde{y}}}{\tilde{y}} = 0 = \frac{\dot{y}}{y} - \frac{\dot{A}}{A} \quad \Rightarrow \quad \frac{\dot{y}}{y} = \frac{\dot{A}}{A} = g
\]

\[
\ln \tilde{y} = \ln Y - \ln A - \ln L \quad \Rightarrow \quad \frac{\dot{\tilde{y}}}{\tilde{y}} = 0 = \frac{\dot{Y}}{Y} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \quad \Rightarrow \quad \frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g + n
\]

<table>
<thead>
<tr>
<th>Measure</th>
<th>Steady State Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{y}, \tilde{k} )</td>
<td>0</td>
</tr>
<tr>
<td>( L )</td>
<td>( n )</td>
</tr>
<tr>
<td>( A, y, k )</td>
<td>( g )</td>
</tr>
<tr>
<td>( Y, K )</td>
<td>( g + n )</td>
</tr>
</tbody>
</table>

**Solving the Model** - try to put capital accumulation equation in terms of \( \tilde{k} \) and \( \tilde{y} \)

Divide by \( K \):

\[
\frac{\dot{K}}{K} = \frac{sY}{K} - \delta
\]

By using the \( \ln \)-differentiate trick on \( \tilde{k} \), we found:

\[
\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}
\]

Solve that for \( \frac{\dot{K}}{K} \) and substitute it into the first equation:

\[
\frac{\dot{\tilde{k}}}{\tilde{k}} + \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = \frac{sY}{K} - \delta
\]

Notice \( \frac{\dot{A}}{A} = g \) and \( \frac{\dot{L}}{L} = n \); multiply \( \frac{sY}{K} \) by \( \frac{AL}{KL} \):

\[
\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{sY}{K} \frac{AL}{KL} - \delta
\]

Use definitions \( \tilde{y} \equiv Y / AL \) and \( \tilde{k} \equiv K / AL \):

\[
\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{s\tilde{y}}{k} - (n + g + \delta)
\]

Solve for \( \tilde{k} \):

\[
\tilde{k} = s\tilde{y} - (n + g + \delta)\tilde{k}
\]

**Steady State** - \( \tilde{k} = 0 \) so

\[
s\tilde{y} = (n + g + \delta)\tilde{k}
\]

Comparative Statics and Transitional Dynamics are similar to original model.