Production Theory

Different From Consumer Theory - easier because no income effects because producers can sell output

    Selling Output - consumers output is happiness or utils which can't be sold; producers can sell their output so it's measurable $\therefore$ it's meaningful to ask if production function is concave (which we couldn't do with utility because concavity is not invariant to transformation); trade-off to this is that we can't transform the production function other than changing units of measurement (e.g., ears of corn, bushels of corn, tons of corn)

Euler's Theorem - if F is <u>homogeneous of degree</u> t (i.e., $F(a\mathbf{x}) = a^t F(\mathbf{x})$), we can take the total derivative with respect to a and evaluate it at $a = 1$

$$\sum_{j=1}^{n} x_j \frac{\partial F}{\partial x_j} = tF(x) \quad \text{(left-hand side is chain rule)}$$

    Applied to Production -

        Economies of Scale - if $F(\mathbf{x})$ is the production function, t determines whether there are economies or diseconomies of scale (i.e., increasing or decreasing returns); $t < 1 \Rightarrow$ diseconomies; $t = 1 \Rightarrow$ constant returns; $t > 1 \Rightarrow$ economies of scale

        Marginal Product - $\partial F/\partial x_j$ = marginal product of resource x_j

        Paying Marginal Product - there used to be debates on whether capitalism could work because people thought if firms paid for resources at their marginal product, the firms would go out of business

Technology - primitive reference similar to preferences; rules that tell firm how it can take inputs and transform them; says which vectors of input and outputs are feasible

    Technology Set - Y, all feasible input/output vectors; can be different in different industries (e.g., standard technology available to all firms or specialized so each firm has different technology set)

    Netput Vector - $\mathbf{y}$ with components that are > 0 for outputs and < 0 for inputs; better for general cases

    Specific Vector - $\mathbf{y(q,x)}$ where $\mathbf{q}$ is set of outputs and $\mathbf{x}$ is set of inputs; all components are positive; better for specific cases where inputs and outputs are known; Note: $\mathbf{y(q,-x)}$ is a netput vector so it's easy to go back and forth between netput and specific output vectors
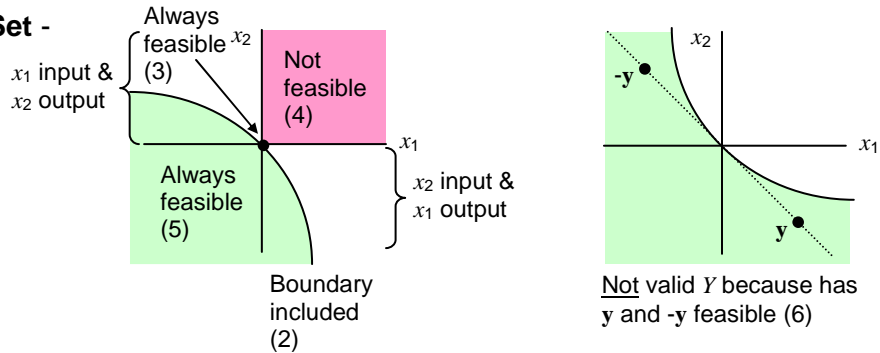
    Feasible Production Plan - y $\in$ Y

Assumptions on Y

1. **Nonempty** - something is feasible (other than $\mathbf{0}$)
2. **Closed** -
3. **Inactivity** - $\mathbf{0} \in Y$; always have possibility of inactivity (producing nothing)
4. **No Free Lunch** - if $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} \notin Y$; that is, if all elements of a netput vector are either zero or positive (i.e., no inputs), the vector is not part of the production set; can't product something from nothing
5. **Free Disposal** - monotonicity assumption; $\mathbf{y} \in Y$ and $\mathbf{y'} \leq \mathbf{y}$, then $\mathbf{y'} \in Y$; Note: $\mathbf{y'} \leq \mathbf{y}$ means every component of $\mathbf{y'}$ is less than the respective components of $\mathbf{y}$ (i.e., produce less with more)
    - **Not Always True** - garbage; "too many cooks spoil the broth"; too much input can be inefficient; it may still be technically feasible if you can set the extra input aside, but that's not always possible

6. **Irreversibility** - $\mathbf{y} \in Y$ and $\mathbf{y} \neq 0$, then $-\mathbf{y} \notin Y$; can't exactly reverse production process because there's always some loss
7. **Convex** -

Drawing Technology Set -

Always feasible x_2 (3)
x_1 input & x_2 output
Not feasible (4)
x_1
x_2 input & x_1 output
Always feasible (5)
Boundary included (2)

x_2
$-\mathbf{y}$
x_1
$\mathbf{y}$
Not valid Y because has $\mathbf{y}$ and $-\mathbf{y}$ feasible (6)

Input Requirement Set

Input Requirement Set - using specific vector, $\mathbf{y}(\mathbf{q},\mathbf{x})$ (instead of netput vector); fix output vector $\mathbf{q}$ and find all input vectors $\mathbf{x}$ that make a feasible production plan; analogous to $R^{\geq}(\mathbf{x})$ from consumer theory
$$V(\mathbf{q}) \equiv \{ \mathbf{x} : \mathbf{y}(\mathbf{q},\mathbf{x}) \in Y \}$$

Properties of $V(\mathbf{q})$
1. **Closed** - weaker than Y closed; Y closed $\Rightarrow V(\mathbf{q})$ closed, but $V(\mathbf{q})$ closed $\not\Rightarrow Y$ closed
2. **Inactivity** - $\mathbf{0} \in V(\mathbf{0})$
3. **No Free Lunch** - $\mathbf{0} \notin V(\mathbf{q})$ for $\mathbf{q} > \mathbf{0}$
4. **Free Disposal** - $\mathbf{q}' > \mathbf{q}'' \Rightarrow V(\mathbf{q}') \subset V(\mathbf{q}'')$; anything that allows us to produce $\mathbf{q}'$ allows us to make $\mathbf{q}''$
5. **Convex** - weaker than Y convex; Y convex $\Rightarrow V(\mathbf{q})$ convex, but $V(\mathbf{q})$ convex $\not\Rightarrow Y$ convex; analogous to $U(\mathbf{x})$ being quasiconcave in consumer theory
   Proof:
   $(\mathbf{q},\mathbf{x}')$ and $(\mathbf{q},\mathbf{x}'') \in Y$ and Y convex $\Rightarrow$
      (1) $(\lambda\mathbf{q} + (1 - \lambda)\mathbf{q}, \lambda\mathbf{x}' + (1 - \lambda)\mathbf{x}'') \in Y$ (definition of convex)
      (2) $\mathbf{x}'$ and $\mathbf{x}'' \in V(\mathbf{q})$ (definition of $V(\mathbf{q})$)
   $(\lambda\mathbf{q} + (1 - \lambda)\mathbf{q}, \lambda\mathbf{x}' + (1 - \lambda)\mathbf{x}'') = (\mathbf{q}, \lambda\mathbf{x}' + (1 - \lambda)\mathbf{x}'') \Rightarrow \lambda\mathbf{x}' + (1 - \lambda)\mathbf{x}'' \in V(\mathbf{q})$
   $\therefore V(\mathbf{q})$ is convex

   $V(\mathbf{q})$ convex
   x_2
   x_1
   q

   Now look at $V(q)$ is convex (using single q and x to make graphs easier); case A shows Y (green area ▢) convex, but B shows Y not convex; in both cases $V(q)$ is an interval (which is a convex set); we can tell A and B apart by looking at returns to scale
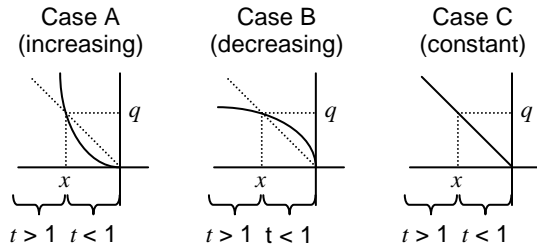

   Case A: Y convex
   $V(q)$
   q
   x

   Case B: Y not convex
   $V(q)$
   q
   x
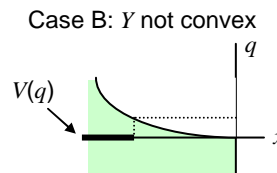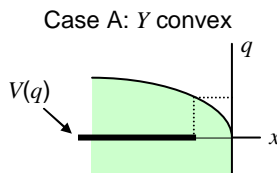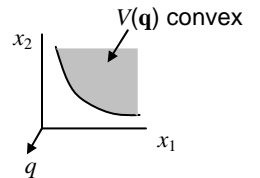
Returns to Scale

$(\mathbf{q},\mathbf{x}) \in Y$, consider whether $(t\mathbf{q},t\mathbf{x}) \in Y$
    $t > 1 \Rightarrow$ A, no; B, yes; C, yes
    $t < 1 \Rightarrow$ A, yes; B, no; C, yes

Increasing RTS - double inputs more than doubles output (for any $t > 1$); notice that using $t < 1$ results in infeasible production so cutting inputs in half means half of output is infeasible
    Why - division of labor and specialization

Decreasing RTS - double inputs less than doubles outputs; if we cut inputs by any percentage, output will be reduced by a smaller percentage
    Why - large bureaucracy; lots of resources used for auditing and accounting

**Constant RTS** - any multiple of inputs will yield the same multiple of output
    Increasing RTS Debates -
        Theorists - used to argue that decreasing returns were impossible; if we observe decreasing RTS it's because we left out some input
            Plant Example - is we have a plant producing some level of output, we should be able to build an exact replica (i.e., double everything) and get twice the output; if that's not the case, then there's something about the new plant that isn't exactly the same as the old plant (land quality, worker quality, management quality, etc.)... but knowing that there's a hidden input isn't very useful
   **Pipe Example** - theorists argue for increasing returns by using a pipe example; area (πr^2) tells how much stuff we can push through pipe; circumference ($2\pi r$) determines how much material is needed to build the pipe; $\therefore$ increasing size of pipe is linear with r, but output increases with r^2; note that at certain point, this doesn't work so we just build a new pipe and move to constant returns
        Empirical - some industries can have increasing returns; determining RTS is "clearly an empirical question" (that's from Slutsky... an theorist); an "uninformed economist's take" (i.e., Slutsky reading abstracts only), most common pattern has increasing RTS initially, then roughly constant RTS for long time; there's the possibility of decreasing RTS afterwards, but empiricists haven't really studied it; it could be because at the point were decreasing RTS occur, a new firm enters to benefit from increasing RTS with a smaller scale (or there's new technology)... this could be a long-run vs. short-run issue
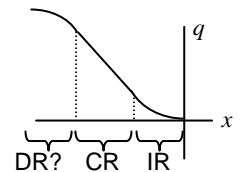
Convex Y - implies that we <u>cannot have increasing</u> returns to scale (in addition to $V(\mathbf{q})$ being convex); this is a convenient assumption for general equilibrium analysis we'll do, but it lacks realism
    Setup Costs - another example that violates convexity of Y; setup cost is limit of increasing returns to scale
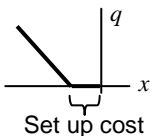
Production Function

Efficient - producing on the boundary of Y

Transformation Surface - also called transformation boundary or transformation frontier; a function that determines whether $\mathbf{y}$ is feasible because $F(\mathbf{y}) \leq 0$ is feasible ($F(\mathbf{y}) > 0$ is not); $\therefore$ surface is set of points that are just feasible: $F(\mathbf{y}) = F(\mathbf{q},\mathbf{x}) = 0$

Production Function - if there is a single output, we can solve $F(q,\mathbf{x})$ for q to get production function: $f(\mathbf{x}) = q$; analogous to utility function in consumer theory (except output is measurable unlike utility); this is typical way we look at technology
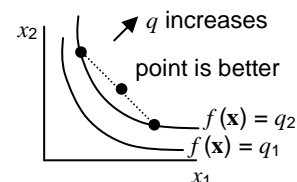
Properties of $f(\mathbf{x})$
1. Defined $\forall\ \mathbf{x} \geq \mathbf{0}$
2. **Continuous** - follows from Y closed
3. **Inactivity** - $f(\mathbf{0}) = 0$
4. **No Free Lunch** - $f(\mathbf{x}) > 0 \Rightarrow \mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ (there has to be some input being used)
5. **Free Disposal** - $f(\mathbf{x})$ nondecreasing in x_j; $\partial f / \partial x_j \geq 0$; form of monotonicity
6. **Quasiconcave** - from $V(q)$ convex
   6a. **Concave** - from Y convex

Deviations from Consumer Theory - properties 3 and 6a were not possible with utility functions because we couldn't use numerical values (3) or because we could use transformations (6a)

Constant Returns - consider $f(\mathbf{x})$ that doesn't have constant RTS (i.e., either increasing or decreasing RTS); we could identify some new function $g(\mathbf{x},m) = mf(\mathbf{x}/m)$; then note that $g(t\mathbf{x},tm) = tmf(t\mathbf{x}/tm) = t(mf(\mathbf{x}/m)) = tg(\mathbf{x},m)$ $\therefore$ $g(\mathbf{x},m)$ has constant returns in $\mathbf{x}$ and m; m is the resource that we left out of $f(\mathbf{x})$

Isoquants

Isoquant - level curve of production function, $f(\mathbf{x}) = $ constant, i.e., $\{\mathbf{x}: f(\mathbf{x}) = q\}$ also boundary of input requirement set, $V(q)$; analogous to indifference curves in consumer theory

Properties
1. Defined $\forall\ \mathbf{x} \geq \mathbf{0}$ (i.e., goes through each input vector $\mathbf{x}$)
2. **Continuous** - follows from Y closed
3. **Monotonicity** - downward sloping, thin lines; stronger than monotonicity of $f(\mathbf{x})$ or $V(q)$
4. **Non-Intersecting** - follows from $f(\mathbf{x})$ being implicitly transitive (#'s are transitive)
5. **Convex to Origin** - follows from $V(q)$ being convex

Cost Function

Engineering Efficiency - using production function assumes engineering efficiency (being technically efficient); economists don't know what this is so we just assume firms know what it is and operate there... assuming you're on the boundary of the technology set

Economic Efficiency - assuming you're <u>at the right point</u> on the boundary of the technology set; i.e., cheapest cost input bundle to produce given output... results in optimization

Optimization Problem - given set output q, we want to minimize resource cost:
Min $\mathbf{w} \cdot \mathbf{x}$ s.t. $f(\mathbf{x}) \geq q$ and $\mathbf{x} \geq \mathbf{0}$

Cost Function - for given input prices ($\mathbf{w}$), cost function gives us the minimum cost required to produce a given output (q); analogous to expenditure function in consumer theory (with $\mathbf{w}$ analogous to $\mathbf{P}$ and q analogous to u); $C(q, \mathbf{w}) \equiv \underset{\mathbf{x}}{\text{Min}} \; \mathbf{w} \cdot \mathbf{x}$ s.t. $f(\mathbf{x}) \geq q$ and $\mathbf{x} \geq \mathbf{0}$

Properties

1. Defined $\forall \; \mathbf{w} > \mathbf{0}$ and $q \geq 0$
2. **Continuous** in q and $\mathbf{w}$
3. **Homogeneous °1 in $\mathbf{w}$** - $C(q, t\mathbf{w}) = tC(q, \mathbf{w})$; doubling input prices doubles cost; has nothing to do with the production function ($f(\mathbf{x})$), just fact that input prices enter linearly in the objective function
4. **Shepherd's Lemma** - look at lagrangian: $C(q, \mathbf{w}) = L(\mathbf{x}^*, \lambda^*, q, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x}^* - \lambda^*(f(\mathbf{x}^*) - q)$
   Increasing in q - $\partial C / \partial q = \lambda > 0$ (**Note:** $\lambda \geq 0$, but we know we're on the production function so we can rule out $\lambda = 0$)
   Nondecreasing in $\mathbf{w}$ - $\partial C / \partial w_j = x_j \geq 0$... both of these are just applications of envelope theorem
5. **Concave in $\mathbf{w}$** -
   <u>Proof</u>:
   Pick any $\mathbf{w}'$, $\mathbf{w}''$ and $t \in (0,1)$
   $C(q, t\mathbf{w}' + (1 - t)\mathbf{w}'') = (t\mathbf{w}' + (1 - t)\mathbf{w}'') \cdot \mathbf{x}^*$... by definition of cost function; label $\mathbf{x}^*$ as function of quantity, q, and input prices, $t\mathbf{w}' + (1 - t)\mathbf{w}''$... $\mathbf{x}^*(q, t\mathbf{w}' + (1 - t)\mathbf{w}'')$, which I'll refrain from doing to save space, but will use for other input vectors
   Because of linearity, we can write $(t\mathbf{w}' + (1 - t)\mathbf{w}'') \cdot \mathbf{x}^* = t\mathbf{w}' \cdot \mathbf{x}^* + (1 - t)\mathbf{w}'' \cdot \mathbf{x}^*$
   We can now look at the cost function for the price vectors individually and find the optimal input vectors $\mathbf{x}(q, \mathbf{w}')$ and $\mathbf{x}(q, \mathbf{w}'')$; by definition of cost function, these input vectors minimize the cost so we know
   $t\mathbf{w}' \cdot \mathbf{x}^* + (1 - t)\mathbf{w}'' \cdot \mathbf{x}^* \geq t\mathbf{w}' \cdot \mathbf{x}(q, \mathbf{w}') + (1 - t)\mathbf{w}'' \cdot \mathbf{x}(q, \mathbf{w}'') = tC(q, \mathbf{w}') + (1 - t)C(q, \mathbf{w}'')$
   $\therefore C(q, t\mathbf{w}' + (1 - t)\mathbf{w}'') \geq tC(q, \mathbf{w}') + (1 - t)C(q, \mathbf{w}'')$ which means $C(q, \mathbf{w})$ is concave in $\mathbf{w}$
   Significance - hessian of $C(q, \mathbf{w})$ is negative semi-definite which we'll combine with Shepherd's Lemma to derive comparative statics on $\partial x_i / \partial w_j$ because second derivative of $C(q, \mathbf{w})$ is $\partial^2 C(q, \mathbf{w}) / \partial w_j \partial w_i = \partial x_i / \partial w_j$

Cost Minimizing Input Demands

Cost Minimizing Input Demand - input vector that minimizes cost of given level of output (q) and specified input prices ($\mathbf{w}$); i.e., the solution to the optimization problem that yields the cost function; by Shepherd's Lemma: $x_j(q,\mathbf{w}) = \partial C(q,\mathbf{w})/\partial w_j$
Analogous to compensated demands in consumer theory
Properties
1. **Complete** - defined for all $\mathbf{w} > 0$ and $q \geq 0$
2. **"Sort of" Continuous** - continuous in $\mathbf{w}$ and q (like ordinary and compensated demand, input demand may not be a function so there may be multiple optimal solutions (many $\mathbf{x}$) but it will always be a convex set)
3. **Homogeneous of Degree 0 in w** - $\mathbf{x}(q,t\mathbf{w}) = \mathbf{x}(q,\mathbf{w})$ (tangency point doesn't change)
4. **Nonnegativity** - $\mathbf{x}(q,\mathbf{w}) \geq 0$
5. **Negative Semidefinite** - substitution matrix (all derivatives of $\mathbf{x}(q,\mathbf{w})$ wrt w_j) is symmetric and negative semidefinite... will never be negative definite because $|\mathbf{S}| = |\mathbf{H}_C| = 0$ (derivative of substitution matrix = derivative of hessian of cost function = 0 because of homogeneity)
   Proof:
   Take derivative of $\mathbf{x}(q,t\mathbf{w}) = \mathbf{x}(q,\mathbf{w})$ wrt t and evaluate it at $t = 1$

   $$\frac{\partial x_i}{\partial w_1}\frac{\partial(tw_1)}{\partial t} + \frac{\partial x_i}{\partial w_2}\frac{\partial(tw_2)}{\partial t} + \cdots + \frac{\partial x_i}{\partial w_n}\frac{\partial(tw_n)}{\partial t} = 0 \text{, } i = 1,2,...,n$$

   Note that $\dfrac{\partial(tw_j)}{\partial t} = w_j$ so we have $\dfrac{\partial x_i}{\partial w_1}w_1 + \dfrac{\partial x_i}{\partial w_2}w_2 + \cdots + \dfrac{\partial x_i}{\partial w_n}w_n = 0$

   Using substitution matrix $\mathbf{S} = \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \cdots & \frac{\partial x_1}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial w_1} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix}$, this can be rewritten as $\mathbf{S}\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{0}$

   $\therefore$ $\mathbf{S}$ not linearly independent and $|\mathbf{S}| = 0$
   Redundant Information - don't need to know all the elements of $\mathbf{S}$ in order to find the rest
   2x2 - simplest case; only need to know 1 element (1/4 of information)

   $\begin{bmatrix} a & \frac{\partial x_1}{\partial w_2} \\ b & c \end{bmatrix}$, know that $b = \dfrac{\partial x_2}{\partial w_1} = \dfrac{\partial x_2}{\partial w_1}$ because of symmetry

   Now find a using $aw_1 + \dfrac{\partial x_1}{\partial w_2}w_2 = 0$, solve for $a = \dfrac{\partial x_1}{\partial w_1} = -\dfrac{\partial x_1}{\partial w_2}\dfrac{w_2}{w_1}$

   Similar technique to find c
   General - just need to know all entries above the diagonal (or corresponding entry below); that's $1 + 2 + ... + (n - 1) = n(n - 1)/2$ independent pieces of information (1/2 as $n \to \infty$)
   Lesson - need to be careful when collecting data that all properties of substitution matrix are satisfied
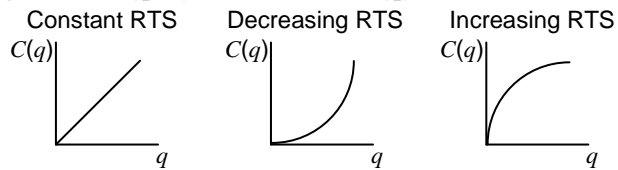
Cost Curves

How do things vary with output for given prices? $C(q,\mathbf{w})$... rewrite as $C(q)$ for convenience;

depends on returns to scale

Constant - $C(q)$ is linear

Decreasing - $C(q)$ is convex

Increasing - $C(q)$ is concave

Constant RTS   Decreasing RTS   Increasing RTS

Average Cost - AC = $C(q)/q$; slope of line from origin to cost curve at q

Marginal Cost - MC = $dC(q)/dq$; slope of cost curve at q
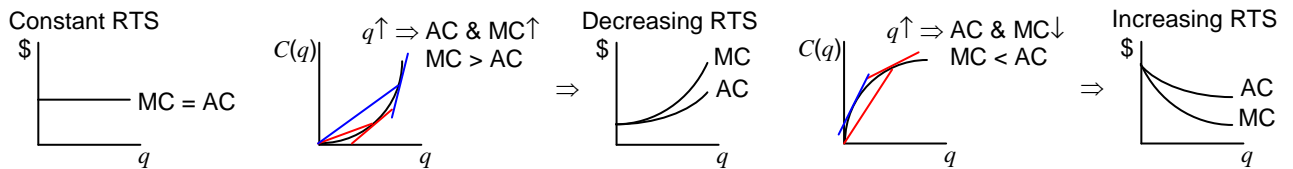
Results - take derivative of AC:

$$\frac{d\text{AC}}{dq} = \frac{dC(q)}{dq}\frac{1}{q} - \frac{C(q)}{q^2} = \frac{1}{q}\left(\frac{dC(q)}{dq} - \frac{C(q)}{q}\right) = \frac{1}{q}(\text{MC} - \text{AC})$$

**Direction of AC** - if MC > AC $\Rightarrow$ $d\text{AC}/dq > 0$ (i.e., AC is increasing); MC < AC $\Rightarrow$ $d\text{AC}/dq < 0$ (i.e., AC is decreasing); MC = AC $\Rightarrow$ $d\text{AC}/dq = 0$ (i.e., MC crosses AC at max and min of AC)

Only Need One - Can solve above for MC = AC + q(dAC/dq); 2 results: (1) only need to know one to find the other; (2) if $q = 0$, MC = AC (as long as $\lim_{q\to 0}|d\text{AC}/dq| \neq \infty$)

Graphically - given results above or looking at graphs above,

Constant RTS $q\uparrow \Rightarrow$ AC & MC$\uparrow$ Decreasing RTS $q\uparrow \Rightarrow$ AC & MC$\downarrow$ Increasing RTS

MC = AC   MC > AC   MC   AC   MC < AC   AC   MC

"Most Reasonable" - given empirical data suggesting firms go through phases of increasing, constant, then decreasing returns to scale, typical graph of AC and MC would look like the one on the left; most text books skip the constant returns and look like the one of the right
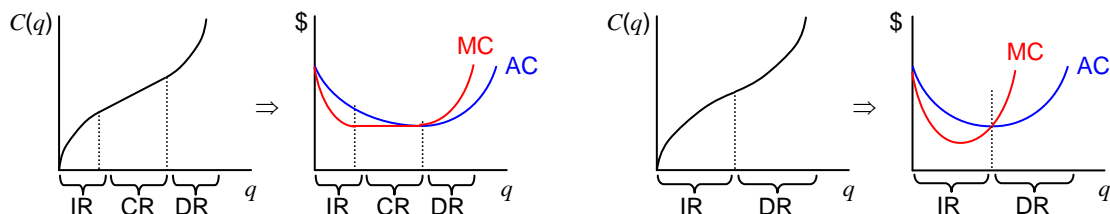
Relationship Between Cost and Production Functions (Duality)

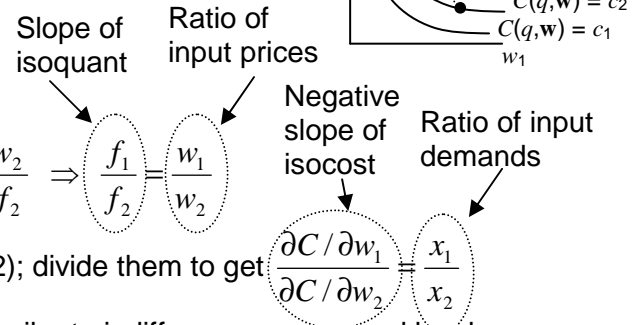**Isocost** - level curve of cost function, $C(q,\mathbf{w})$ = constant; i.e., $\{\mathbf{w}: C(q,\mathbf{w}) = c\}$
Duality - can relate isocost to isoquant...
    Lagrangian - $L = \mathbf{w} \cdot \mathbf{x} - \lambda(f(\mathbf{x}) - q)$
    First Order Conditions -

$$\frac{\partial L}{\partial x_1} = w_1 - \lambda \frac{\partial f(\mathbf{x})}{\partial x_1} = w_1 - \lambda f_1 = 0$$

$$\frac{\partial L}{\partial x_2} = w_2 - \lambda \frac{\partial f(\mathbf{x})}{\partial x_2} = w_2 - \lambda f_2 = 0 \implies \lambda = \frac{w_1}{f_1} = \frac{w_2}{f_2} \implies \left(\frac{f_1}{f_2}\right) = \left(\frac{w_1}{w_2}\right)$$

Slope of isoquant   Ratio of input prices   Negative slope of isocost   Ratio of input demands

    Shepherd's Lemma - lemma says $\partial C/\partial w_j = x_j$; (j = 1, 2); divide them to get $\left(\dfrac{\partial C / \partial w_1}{\partial C / \partial w_2}\right) = \left(\dfrac{x_1}{x_2}\right)$

    Result - inverse elasticities of isocost and isoquant (similar to indifference curves and level curves of indirect utility function in consumer theory) $\therefore$ as one set of curves becomes more straight, the other becomes more curved; extreme case is one of them being straight lines and the other being right angles
Returns to Scale Duality - same relationship between returns to scale for cost function and production function (i.e., the mirror each other... or are "inverses" of each other)
    RTS for Production - already looked at graphs for RTS from cost point of view, need similar way to view it for production using isoquants; label isoquants with new variable t so we get output as function of t: $f(t\mathbf{x}) = g(t)$, how look at graph of $g(t)$ vs. t... this will be similar to graphs of $C(q)$ vs. q only backwards (i.e., $C(q)$ graph is convex for decreasing RTS, so $g(t)$ graph is concave

Decreasing RTS: Convex $C(q)$ vs q; Concave $g(t)$ vs t

Increasing RTS: Concave $C(q)$ vs q; Convex $g(t)$ vs t
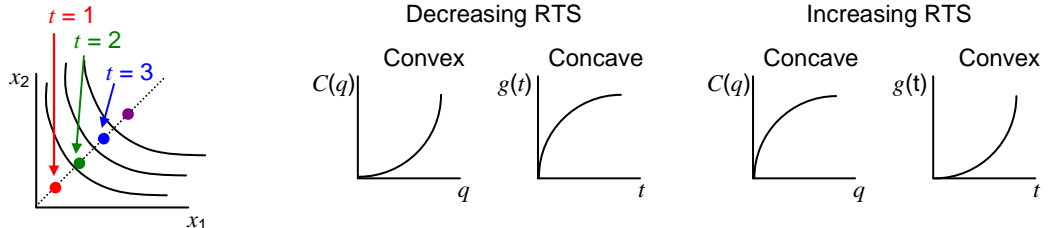
Why Care - it's hard to get data on inputs (some firm's don't even know all their inputs); but we can get data on outputs and costs; we can run a regression on this data to solve for the cost function ($C(q,\mathbf{w})$) which allows us to determine returns to scale

Profit and Supply Curve

Firm's don't operate by minimizing costs, they maximize profit (π)

Profit - total revenue minus total cost; $\pi(\mathbf{q}) = \mathbf{P} \cdot \mathbf{q} - C(\mathbf{q}, \mathbf{w})$

    Single Output - for simplicity look at case with single output: $\underset{q}{\text{Max}}\ Pq - C(q,\mathbf{w})$ s.t. $q \geq 0$

        that's constrained optimization of a single variable

        Interior Solution -

            1st Order - $\dfrac{\partial \pi}{\partial q} = P - \dfrac{\partial C(q,\mathbf{w})}{\partial q} = 0 \ \Rightarrow\ P$ - MC = 0; since P is given, firm adjusts q in

            order to get **MC = P**

            2nd Order - $\dfrac{\partial^2 \pi}{\partial q^2} = -\dfrac{\partial^2 C(q,\mathbf{w})}{\partial q^2} < 0$ (i.e., concave) $\Rightarrow \dfrac{\partial^2 C(q,\mathbf{w})}{\partial q^2} > 0$ so the cost

            function is convex... that's decreasing returns to scale (another reason
            economists don't like increasing returns to scale)

            $\dfrac{\partial^2 C(q,\mathbf{w})}{\partial q^2} = \dfrac{\partial \text{MC}}{\partial q} > 0$ so firms also wants **MC rising**

        Corner Solution - q = 0 $\Rightarrow \dfrac{\partial \pi}{\partial q} = P - \dfrac{\partial C(q,\mathbf{w})}{\partial q} \leq 0$ (Kuhn-Tucker condition)

            That is, the first order condition tells us we want to decrease q, but we're already at
            the limit ($q = 0$); Note: this solution is only valid if $P \leq$ MC at $q = 0$

        Global Max - given two local maxima found above, we evaluate both to determine the
            optimal level of output; we want to pick the local maxima that has the greatest profit:
            $\pi(0) = 0$ and $\pi(q^*) = Pq^* - C(q^*,\mathbf{w})$; $\therefore$ we'd only choose q^* if we know
            $\pi(q^*) = Pq^* - C(q^*,\mathbf{w}) \geq \pi(0) = 0 \ \Rightarrow\ P \geq C(q^*,\mathbf{w})/q^* =$ AC

Supply Curve - graphs solution to optimization problem discussed above:

$q(P,\mathbf{w}) = \begin{cases} 0, \text{ if } P < \text{Min(AC)} \\ q^*, \text{ if } P \geq \text{AC} \end{cases}$ (where q^* is the point at which $P =$ MC and MC is rising)

    Note: at $P = $ Min(AC) the firm is indifferent between producing nothing and
    producing q^*; this is easier to see graphically

Comparative Statics - focus on interior solution

    Ouput Price (P) - totally differentiate first order conditions wrt P; will do this one
        step by step, but skip steps that repeat for input prices later
        1. Write out lagrangian: $L = Pq - C(q,\mathbf{w})$

        2. Take K-T conditions: $q \geq 0 \Rightarrow \dfrac{\partial L}{\partial q} = P - \dfrac{\partial C(q,\mathbf{w})}{\partial q} = 0$

        3. Discard conditions with corner solutions... this is already done since we're assuming in
            interior solution ($q \geq 0$)
        4. Implicitly solve: $q^* = q(P,\mathbf{w})$... that is, q^* is a function of P and $\mathbf{w}$ (parameters of the
            model); now we substitute that into the K-T condition which makes in an identity:

        $\dfrac{\partial L}{\partial q} = P - \dfrac{\partial C(q(P,\mathbf{w}),\mathbf{w})}{\partial q} \equiv 0$

        5. Totally differentiate wrt P: $1 - \dfrac{\partial^2 C}{\partial q^2}\dfrac{dq}{dP} = 0$

        6-7. Matrix form and Cramer's Rule: don't really need this since it's only one equation
            and one unknown; we can solve directly
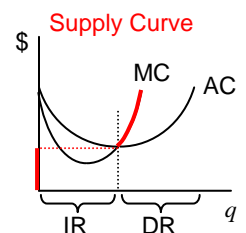
$$\frac{dq}{dP} = \frac{1}{\frac{\partial^2 C}{\partial q^2}}$$

8. Say something interesting: we know $\frac{\partial^2 C}{\partial q^2} > 0$ from second order condition, $\therefore \frac{dq}{dP} > 0$,

    that is, if we increase price, we will increase output ($P\uparrow \Rightarrow q\uparrow$)

Input Price (w_j) - pick up with step 5 from above

5. Totally differentiate wrt w_j: $-\frac{\partial^2 C}{\partial q^2}\frac{dq}{dw_j} - \frac{\partial^2 C}{\partial q \partial w_j} = 0$

6-7. Matrix form and Cramer's Rule: again, don't really need this; we can solve directly

$$\frac{dq}{dw_j} = -\frac{\partial^2 C}{\partial q \partial w_j} \bigg/ \frac{\partial^2 C}{\partial q^2} = 0 \ldots \frac{dq}{dw_j} = -\frac{\partial x_j}{\partial q} \bigg/ \frac{\partial^2 C}{\partial q^2} = 0 \text{ (see below; used on p.13)}$$

8. Say something interesting: we know $\frac{\partial^2 C}{\partial q^2} > 0$ from second order condition, $\therefore$ sign of

 dq/dw_j depends on sign of numerator

    First Attempt - $\frac{\partial^2 C}{\partial q \partial w_j} = \frac{\partial MC}{\partial w_j}$; it seems natural that this would be > 0, but don't

    know for sure so look at another to solve

    Second Attempt - use symmetry: $\frac{\partial^2 C}{\partial q \partial w_j} = \frac{\partial^2 C}{\partial w_j \partial q} = \frac{\partial(\partial C/\partial w_j)}{\partial q} = \frac{\partial x_j}{\partial q}$ (that last

    step used Shepherd's Lemma)

    Expansion Path of the Firm - analogous to income consumption curve; function
        that shows how input mix ($\mathbf{x}$) varies with output level (q); line connecting all
        optimal solutions to cost minimization problem for all levels of q; like ICC, this
        line can bend left or down so $\partial x_j/\partial q < 0$ or it can bend up or right so $\partial x_j/\partial q > 0$
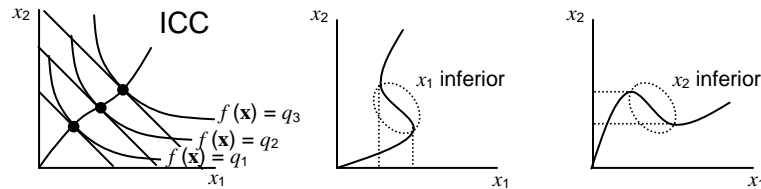
    Normal Input - $\partial x_j/\partial q > 0 \Rightarrow dq/dw_j < 0$, i.e., if price of input goes up, we
        produce less output (that's what we expected with the first attempt)
    Inferior Input - $\partial x_j/\partial q < 0 \Rightarrow dq/dw_j > 0$, i.e., if price of input goes up, we
        produce more output; this seems contrary to intuition, but if you imagine an
        inferior input (say unskilled labor), if the cost of that input rises, firms may
        move away form that particular input to more productive inputs so output
        actually increases
        Note: can never have an input that's always inferior (or firm would never use
            that input); if input is being used, it was normal at one point so inferior
            input is a local phenomenon (depends on q)

Profit Function and Profit Maximizing Supplies & Demands

Switch to netput vector $\mathbf{y}$ (remember $y_j > 0$ is output and $y_j < 0$ is input)

Real World - firm is simultaneously choosing inputs and output(s); $\underset{\mathbf{y}}{\text{Max}}\ \mathbf{P}\cdot\mathbf{y}$ s.t. $F(\mathbf{y}) \leq 0$

    Profit - $\mathbf{P}\cdot\mathbf{y}$; this is revenue minus cost

    Constraint - $F(\mathbf{y})$ is the transformation surface (p.4), anything ≤ 0 is feasible, but we know optimal solution will be at $F(\mathbf{y}) = 0$ to be on the boundary of the technology set

Solvable? - depends on returns to scale

    Increasing RTS - if you can produce a given level, you can always make more (i.e., unbounded if any feasible solution); else solution is zero vector (can't make anything)

        <u>Proof</u>: assume $\mathbf{P}\cdot\hat{\mathbf{y}} > 0$ is the solution; since we have increasing RTS, $\mathbf{P}\cdot t\hat{\mathbf{y}}$ is also feasible for $t > 0$; but because of homogeneity of profit we know $\mathbf{P}\cdot t\hat{\mathbf{y}} = t\mathbf{P}\cdot\hat{\mathbf{y}} > \mathbf{P}\cdot\hat{\mathbf{y}}$, $\therefore \hat{\mathbf{y}}$ couldn't be the solution

    Constant RTS - same argument as increasing RTS applies; only difference is firm producing at $\mathbf{P}\cdot\hat{\mathbf{y}} = 0$ (no profit) at which point the firm is indifferent between producing and not producing... this results in a supply curve that is not well behaved; if $P > $ MC, produce ∞; if $P < $ MC, produce 0; if $P = $ MC, produce any level of output

        Problem - firm can't set q to get MC $= P$ since MC is independent of q

        Solution - market sets P (through demand)

    Decreasing RTS - profit maximization problem has a unique solution; don't need entire technology set to have decreasing RTS, just the region where the solution is $\therefore$ we will assume we have a problem with a solution (i.e. local decreasing RTS)
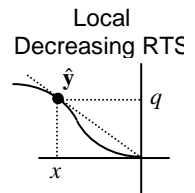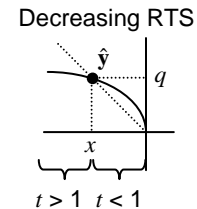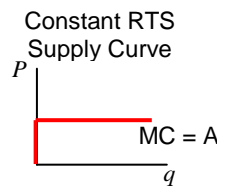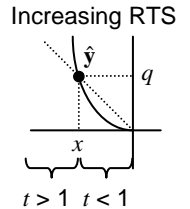
Profit Function - optimized value function; $\pi(\mathbf{P}) \equiv \underset{\mathbf{y}}{\text{Max}}\ \mathbf{P}\cdot\mathbf{y}$ s.t. $F(\mathbf{y}) \leq 0$

Properties

    1. **Complete** - defined for all $\mathbf{P}$
    2. **Continuous** - continuous in $\mathbf{P}$
    3. **Homogeneous of Degree 1 in $\mathbf{P}$** - $\pi(t\mathbf{P}) = t\pi(\mathbf{P})$
        (Note: this is nominal π, real π unchanged)
    4. **Monotonicity** -
        Hotelling's Lemma - apply envelope theorem to $L = \mathbf{P}\cdot\mathbf{y} - F(\mathbf{y})$

$$\frac{\partial \pi(\mathbf{P})}{\partial P_j} = \frac{\partial L(\mathbf{P}, \mathbf{y}^*, \lambda^*)}{\partial P_j} = y_j \quad \text{(only worried about where } \mathbf{P} \text{ enters directly in } L)$$

        $\therefore$ sign depends on whether y_j is input or output so change in profit from change in price depends on how much firm owns (y_j output) or buys (y_j input)... similar to result from generalized Slutsky equation

    5. **Convex** - follows from fact that this is an optimized value function with parameter entering linearly in objective (i.e., not quasi) and it's a maximization problem (convex not concave); $\therefore$ hessian of π is positive semidefinite (all principal minors ≥ 0)

Profit Maximizing Supplies and Demands - $\mathbf{y}(\mathbf{P})$ that solves maximization problem
Properties
1. **Complete** - defined for all $\mathbf{P}$
2. **"Sort of" Continuous** - continuous in $\mathbf{P}$ (like ordinary and compensated demand, profit maximizing supplies & demands may not be a function so there may be multiple optimal solutions (many $\mathbf{y}$) but it will always be a convex set)
3. **Homogeneous of Degree 0 in $\mathbf{P}$** - $\mathbf{y}(t\mathbf{P}) = \mathbf{y}(\mathbf{P})$ (so change in $\mathbf{P}$ doesn't change $\mathbf{y}$)
4. **Monotonicity** - doesn't have any monotonicity property because it combines inputs ($y_j < 0$) and outputs ($y_j > 0$); what we do know is that $\mathbf{y}$ must have some negative components because of no free lunch
5. **"Convexity"** - comparative statics matrix ($\partial y_i / \partial P_j$) is same as hessian of π ($\partial^2 \pi / \partial P_j \partial P_i$)... symmetric and positive semidefinite
    Own Price Effect - $\partial y_j / \partial P_j > 0$; this makes sense if y_j is an output ($P_j \uparrow \Rightarrow y_j \uparrow$), but for inputs we expect $P_j \uparrow \Rightarrow y_j \downarrow$... not so because $y_j < 0$ for inputs $\therefore$ what we expect is $P_j \uparrow \Rightarrow |y_j| \downarrow$ (i.e., $\partial |y_j| / \partial P_j < 0$; $P_j \uparrow$ so we use less input which means $\partial y_j / \partial P_j > 0$

    Think about it... $\partial y_j / \partial P_j > 0$ made sense to me without this explanation of inputs and outputs which I think just gets confusing. Inputs are negative so if their price goes up, you want to use less of them (i.e., make the negative number bigger).

Short-Run vs. Long-Run

Short-Run - some variable(s) fixed
Long-Run - all possible variables can be adjusted; so far all we looked at has been LR
Context - SR & LR mean different things when talking about firms and markets
    Firm - SR & LR address inputs; SR means some input to the firm is fixed
    Market - SR & LR address number of firms; SR means number of firms is fixed
Time - length of time isn't so important because it depends on the industry; for some SR is a
    moth, for others it could be 5 years (orange groves mature in 15 years!)

Simple Version

Will simply look at holding output constant for short-run to get a feel for comparing short-run and
    long-run (we'll come back and do it the real way later)

    Long-Run - $\underset{\mathbf{q,\,x}}{\text{Max}}\ \mathbf{P \cdot q} - \mathbf{w \cdot x}$ **Short-Run** - $\underset{\mathbf{q,\,x}}{\text{Max}}\ \mathbf{P \cdot q} - \mathbf{w \cdot x}$

           s.t. $F(\mathbf{q,x}) = 0$ s.t. $F(\mathbf{q,x}) = 0$ and $\mathbf{q} = \overline{\mathbf{q}}$

    Note: short-run problem basically is cost minimization because $\Delta \mathbf{P}$ has no effect on $\mathbf{x}$

Profit Maximizing Input Demand - $\mathbf{x(P,w)}$, part of solution to long-run profit max problem
Profit Maximizing Output Supply - $\mathbf{q(P,w)}$, part of solution to long-run profit max problem
Cost Minimizing Input Demand - $\mathbf{\hat{x}(\overline{q}, w)}$, solution to short-run cost minimization problem; this
    is the input demand we saw before (p. 6)
Relating Demands - by setting $\overline{\mathbf{q}} = \mathbf{q(P,w)}$ the profit maximizing and cost minimizing input

    demands are identical: $\mathbf{x(P, w)} \equiv \mathbf{\hat{x}(q(P, w), w)}$

    Own Price Effect - look at good j and totally differentiate wrt w_j (own price)

      Δ profit max. $\dfrac{\partial x_j}{\partial w_j} = \underbrace{\dfrac{\partial \hat{x}}{\partial q}\dfrac{\partial q}{\partial w_j}}_{\substack{\text{Output}\\\text{Effect}}} + \underbrace{\dfrac{\partial \hat{x}}{\partial w_j}}_{\substack{\text{Substitution}\\\text{Effect}}} < 0$ Similar to Slutsky equation from
      input demand   consumer theory, but not ambiguous
      from Δ own price

    Substitution Effect - from p.6: $\dfrac{\partial \hat{x}}{\partial w_j} < 0$ (diagonal element of hessian of $C(\mathbf{q,w})$, -semidef);

    given same level of output, if the price of one input goes up, we're likely to substitute
    another input for it

    Output Effect - $\dfrac{\partial \hat{x}}{\partial q}\dfrac{\partial q}{\partial w_j} = \dfrac{\partial \hat{x}}{\partial q}\left(-\dfrac{\partial \hat{x}_j / \partial q}{\partial^2 C / \partial q^2} \right)$ (substitution from p.10)

    $= -\dfrac{(\partial \hat{x}_j / \partial q)^2}{\partial^2 C / \partial q^2}$,

    this is < 0 because the numerator is > 0 (squared term) and the
    denominator is > 0 (from second order condition); if we're free to
    adjust output, a rise in input price will force us to produce less
    output overall (similar to income effect in consumer theory)

    Result - $\dfrac{\partial x_j}{\partial w_j} < 0$ (as expected since we know that $\dfrac{\partial y_j}{\partial P_j} > 0$ [p.12]... $x_j = -y_j$ and $w_j = P_j$)
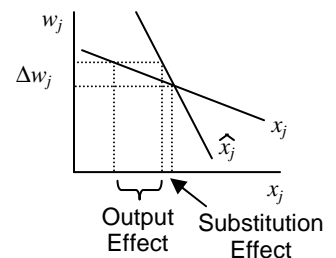
LeChatelier Principle - result from thermodynamics; add a constraint to an optimization problem and the choice variable will be less responsive to changes in its own parameters in the objective function... for this problem that means

$$\left|\frac{\partial x_j}{\partial w_j}\right| \geq \left|\frac{\partial \hat{x}_j}{\partial w_j}\right|$$

Real Version

Now we'll look at a simple, but real version of long-run vs. short-run... cost minimization (rather than full blown profit maximization)

Long-Run - $\underset{\mathbf{x}}{\text{Min}} \ \mathbf{w} \cdot \mathbf{x}$ $\qquad\qquad$ **Short-Run** - $\underset{\mathbf{x}^F, \mathbf{x}^V}{\text{Min}} \ \mathbf{w}^F \cdot \mathbf{x}^F + \mathbf{w}^V \cdot \mathbf{x}^V$

$\qquad\qquad$ s.t. $f(\mathbf{x}) = q$ $\qquad\qquad\qquad\qquad$ s.t. $f(\mathbf{x}^F, \mathbf{x}^V) = q$ and $\mathbf{x}^F = \overline{\mathbf{x}}^F$

Note: now we have fixed and variable costs and inputs in short-run

Fixed Cost - don't change wrt q; $FC = \mathbf{w}^F \cdot \mathbf{x}^F$
  Sunk Cost - fixed cost that has to be paid even if $q = 0$ (e.g., fee to enter industry)
  Overhead Cost - fixed cost that can only be avoided if $q = 0$, as soon as $q > 0$ the overhead cost has to be paid no matter how big or small q is (e.g., setup cost for production line)
  Teaching Example - building is sunk cost because it's paid for even if there are no students; professor is overhead cost because you only need to pay for the professor if there are students
  Movie Example - film is sunk cost because it's paid for even if theater doesn't show it; projectionist is overhead cost because you only need to hire one if you show the movie; it doesn't matter if you sell 1 ticket or 600, the cost for the projectionist is the same
Variable Cost - change wrt q; $VC(q, \mathbf{w}^V, \mathbf{x}^F)$... function of output, variable input costs and level of fixed inputs (which determine level of variable inputs)
Short-Run Cost Function - $C(q, \mathbf{w}^V, \mathbf{w}^F, \mathbf{x}^F) = VC(q, \mathbf{w}^V, \mathbf{x}^F) + FC$
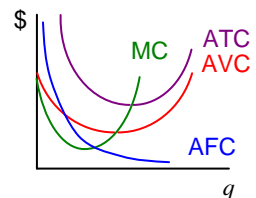
Average Costs
  Average Total Cost - ATC = C/q $\qquad\qquad$ ATC = AVC + AFC
  Average Variable Cost - AVC = VC/q
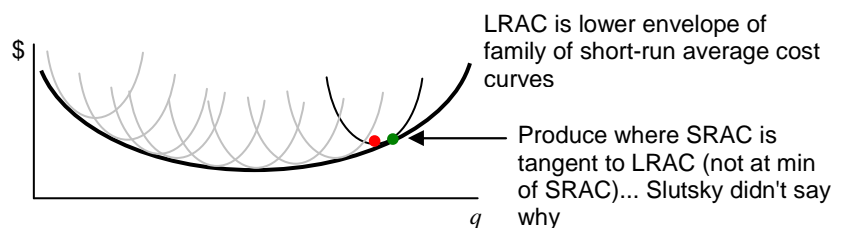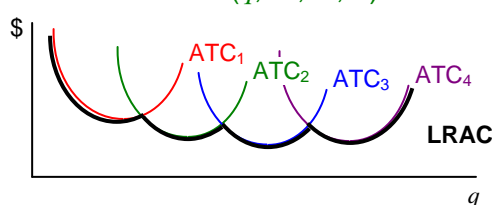 **Average Fixed Cost** - AFC = FC/q

Marginal Cost
  MC = dC/dq = dVC/dq (because dFC/dq = 0) $\therefore$ MC goes through the min of ATC and AVC (which we can prove by taking derivative of ATC or AVC like we did on p.7); second result we had is also valid here: "if $q = 0$, MC = AC (as long as $\lim_{q \to 0}|dAC/dq| \neq \infty$)"
    Note: this rule only holds for AVC because $\lim_{q \to 0}|dATC/dq| = \infty$) $\therefore$ if $q = 0$, MC = AVC
U-Shaped - from law of diminishing marginal returns (extra output from additional input gets smaller) $\therefore$ now we have u-shaped curves without having to appeal to decreasing returns to scale (more realistic)

Long-Run Average Cost - minimum of all possible short-run average cost curves;
  LRAC = Min ATC($q, \mathbf{w}^V, \mathbf{w}^F, \mathbf{x}^F$)

LRAC is lower envelope of family of short-run average cost curves

Produce where SRAC is tangent to LRAC (not at min of SRAC)... Slutsky didn't say why

Relating SR and LR - look at cost minimizing input demands

Short-Run - $\hat{\mathbf{x}}^V(q,\mathbf{w}^V,\mathbf{x}^F)$

Long-Run - $\mathbf{x}^V(q,\mathbf{w}^V,\mathbf{w}^F)$, $\mathbf{x}^F(q,\mathbf{w}^V,\mathbf{w}^F)$... don't really have fixed and variable in long-run, but breaking them up so we can compare them to the short-run

Relating Demands - just like p.13... determine optimal fixed input in long run and plug it into short-run input demand at it'll be the same as the optimal long-run input demand

$\mathbf{x}^V(q,\mathbf{w}^V,\mathbf{w}^F) \equiv \hat{\mathbf{x}}^V(q,\mathbf{w}^V,\mathbf{x}^F(q,\mathbf{w}^V,\mathbf{w}^F))$

From here we'd totally differentiate x_j^V with respect to w_j^V and compare short and long run input demands and get something similar to the Slutsky equation (like we did on p.13); By LeChatelier Principle, we'd find that input depends are more responsive in the long-run than in the short-run

$$\left|\frac{\partial x_j^V}{\partial w_j^V}\right| \geq \left|\frac{\partial \hat{x}_j^V}{\partial w_j^V}\right|$$

At q^* LRMC = SRMC and LRMC is flatter than SRMC
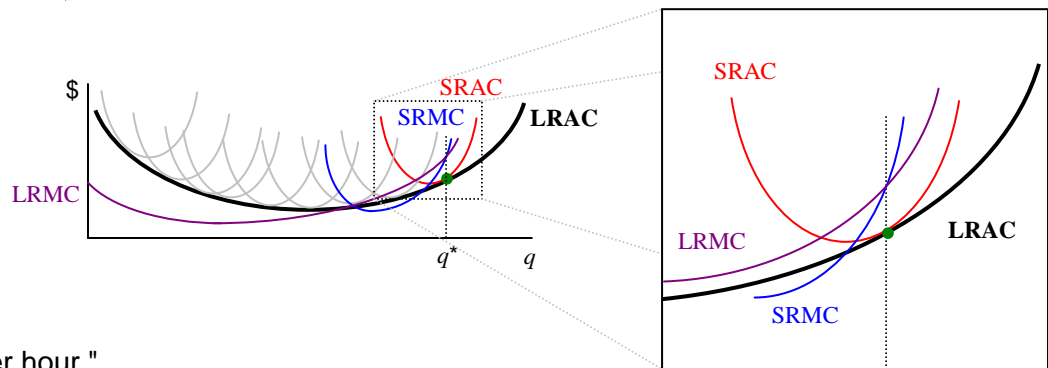
"Needed another hour."

Production - Summary

Summary of Properties -

Technology Set - Y	Input Requirement Set - $V(\mathbf{q})$
[1] Nonempty: some $\mathbf{y} \in Y$ with $\mathbf{y} \neq \mathbf{0}$	$V(\mathbf{q}) \equiv \{\ \mathbf{x} : \mathbf{y}(\mathbf{q},\mathbf{x}) \in Y\ \}$
[2] Closed	Analogous to $R^{\geq}(\mathbf{x})$ (consumer theory)
[3] Inactivity: $\mathbf{0} \in Y$	Closed [2]
[4] No Free Lunch: $\mathbf{y} \geq \mathbf{0}$ & $\mathbf{y} \neq \mathbf{0} \Rightarrow \mathbf{y} \notin Y$	Inactivity: $\mathbf{0} \in V(\mathbf{0})$ [3]
$\quad$ Can't have all outputs with no inputs	No Free Lunch: $\mathbf{0} \notin V(\mathbf{q})$ for $\mathbf{q} > \mathbf{0}$ [4]
[5] Free Disposal: $\mathbf{y} \in Y$ & $\mathbf{y}' \leq \mathbf{y} \Rightarrow \mathbf{y}' \in Y$	Free Disposal: $\mathbf{q}' > \mathbf{q}'' \Rightarrow V(\mathbf{q}') \subset V(\mathbf{q}'')$ [5]
$\quad$ Monotonicity; produce less (or same) with more	Convex: analogous to $U(\mathbf{x})$ quasiconcave [7]
[6] Irreversibility: $\mathbf{y} \in Y$ & $\mathbf{y} \neq 0 \Rightarrow -\mathbf{y} \notin Y$	
$\quad$ Will be lose if we reverse production process	
[7] Convex $\Rightarrow$ no increasing returns to scale	

Production Function - $f(\mathbf{x})$	Isoquant - IQ(q)
$f(\mathbf{x}) = q \Leftrightarrow F(q,\mathbf{x}) = 0$	$\text{IQ}(q) \equiv \{\mathbf{x} : f(\mathbf{x}) = q\}$ (level curves of $f(\mathbf{x})$)
Analogous to $U(\mathbf{x})$ (consumer theory)	Analogous to indifference curves
Defined $\forall\ \mathbf{x} \geq \mathbf{0}$	Goes through each input vector $\mathbf{x}$
Continuous [2]	Continuous [2]
Inactivity: $f(\mathbf{0}) = 0$ [3]	"Thin" lines
No Free Lunch: $f(\mathbf{x}) > 0 \Rightarrow \mathbf{x} \geq \mathbf{0}$ & $\mathbf{x} \neq \mathbf{0}$ [4]	Downward sloping
Free Disposal: $f(\mathbf{x})$ nondecreasing in x_j [5]	Non-intersecting
$\quad$ Monotonicity; $\partial f / \partial x_j \geq 0$	Convex to origin: if $V(q)$ is convex
Quasiconcave: if $V(q)$ is convex	
Concave: if Y is convex [7]	

Cost Function - $C(q,\mathbf{w})$	Input Demands - $\mathbf{x}(q,\mathbf{w})$
$C(q,\mathbf{w}) \equiv \text{Min } \mathbf{w} \cdot \mathbf{x} \text{ st } f(\mathbf{x}) \geq q \text{ \& } \mathbf{x} \geq \mathbf{0}$	$x_j(q,\mathbf{w}) = \partial C(q,\mathbf{w})/\partial w_j$
$\equiv L(\mathbf{x}^*,\lambda^*,q,\mathbf{w}) = \mathbf{w} \cdot \mathbf{x}^* - \lambda^*(f(\mathbf{x}^*) - q)$	Analogous to $\mathbf{x}^c(\mathbf{P}, u)$ (consumer theory)
Analogous to $E(\mathbf{P},u)$ (consumer theory)	Defined $\forall$ $\mathbf{w} > \mathbf{0}$ and $q \geq 0$
Defined $\forall$ $\mathbf{w} > \mathbf{0}$ and $q \geq 0$	"Sort of" continuous
Continuous in q and $\mathbf{w}$	Homogeneous $°0$ in $\mathbf{w}$: $\mathbf{x}(q,t\mathbf{w}) = \mathbf{x}(q,\mathbf{w})$
Homogeneous $°1$ in $\mathbf{w}$: $C(q,t\mathbf{w}) = tC(q,\mathbf{w})$	Nonnegative: $\mathbf{x}(q,\mathbf{w}) \geq 0$
Increasing in q: $\partial C/\partial q = \lambda > 0$	Substitution matrix negative semidefinite
Non-decreasing in w_j: $\partial C/\partial w_j = x_j \geq 0$	and symmetric
Concave in $\mathbf{w}$	

Profit Function - $\pi(\mathbf{P})$	Profit Max S & D - $\mathbf{y}(\mathbf{P})$
$\pi(\mathbf{P}) \equiv \text{Max } \mathbf{P} \cdot \mathbf{y} \text{ st } F(\mathbf{y}) \leq 0$	$y_j(\mathbf{P}) = \partial \pi(\mathbf{P})/\partial P_j$
$\equiv L(\mathbf{P},\mathbf{y}^*,\lambda^*) = \mathbf{P} \cdot \mathbf{y}^* - \lambda^* F(\mathbf{y}^*)$	Defined $\forall$ $\mathbf{P}$
Defined $\forall$ $\mathbf{P}$	"Sort of" continuous
Continuous in $\mathbf{P}$	Homogeneous $°0$ in $\mathbf{P}$: $\mathbf{y}(t\mathbf{P}) = \mathbf{y}(\mathbf{P})$
Homogeneous $°1$ in $\mathbf{P}$: $\pi(t\mathbf{P}) = t\pi(\mathbf{P})$	Monotonicity: $\mathbf{y} \neq \mathbf{0}$, $\exists$ $y_j < 0$ (no free lunch)
Monotonicity: Hotelling's Lemma	Substitution matrix positive semidefinite
$\partial\pi/\partial P_j = y_j > 0$ if y_j is output; < 0 for input	and symmetric
Convex in $\mathbf{P}$	