Consumer Theory - Indirect Utility Function

Indirect Utility Function - $V(\mathbf{P},I) \equiv \text{Max } U(\mathbf{x}) \text{ st } \mathbf{P}\cdot\mathbf{x} \le I \text{ and } \mathbf{x} \ge \mathbf{0}$; optimized value function (i.e., solve the maximization problem, then plug solution back into $U(\mathbf{x})$ to get $V(\mathbf{P},I)$); lists the solutions to the maximization problem for the various values of the parameters $\mathbf{P}$ and I

With Lagrangian - to simplify notation let $\mathbf{x}^* = \mathbf{x}(\mathbf{P},I)$ and $\lambda^* = \lambda(\mathbf{P},I)$... the solutions to the maximization problem as functions of $\mathbf{P}$ and I (using a little macro notation, using same symbol for the value and the function); now $V(\mathbf{P},I) \equiv L(\mathbf{x}^*,\lambda^*,\mathbf{P},I) = U(\mathbf{x}^*) - \lambda^*(\mathbf{P}\cdot\mathbf{x}^* - I)$

Because $\mathbf{x}^*$ & λ^* will be optimal for a given $\mathbf{P}$ an I, this portion will be zero

Use Envelope Theorem - look at changes in $\mathbf{P}$ and I (ignoring $\mathbf{x}^*$ and λ^*)

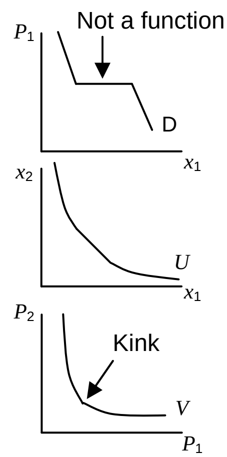
$$\frac{\partial V}{\partial I} = \left.\frac{\partial L}{\partial I}\right)^{\mathbf{x}^*,\lambda^*\text{ fixed}} = \lambda^*$$ (derivative of L wrt I, ignoring were I enters $\mathbf{x}^*$ and λ^*)

$$\frac{\partial V}{\partial P_j} = \left.\frac{\partial L}{\partial P_j}\right)^{\mathbf{x}^*,\lambda^*\text{ fixed}} = -\lambda^* x_j^*$$

Roy's Identity - can manipulate the equalities above to set them equal to λ^*, then solve for x_j^*; thus you can get demands from the indirect utility function:

$$x_j(\mathbf{P},I) = -\frac{\partial V/\partial P_j}{\partial V/\partial I}$$

Properties - of indirect utility function $V(\mathbf{P},I)$ (assuming $U(\mathbf{x})$ satisfies its properties)



1. **Complete** - defined for all $\mathbf{P} > 0$ and I
2. **Continuous** - continuous in $\mathbf{P}$ and I (even if demands aren't)
   $V(\mathbf{P},I) = U(\mathbf{x}(\mathbf{P},I))$; $\mathbf{x}(\mathbf{P},I)$ may not be a function, but those places still have the same utility so V is continuous (although will have a kink or corner point)
3. **Homogeneous of Degree 0 in Prices and Income** - proportional changes in prices and income have no effect; $V(t\mathbf{P},tI) = V(\mathbf{P},I)$
4. **Monotonic** - increasing in I and non-increasing in P_j
   $$\frac{\partial V}{\partial I} = \lambda^* > 0;\ \frac{\partial V}{\partial P_j} = -\lambda^* x_j^* \le 0$$
5. **Quasiconvex** in $\mathbf{P}$ and I; **Note:** because of homogeneity, $|\mathbf{BH}| = 0$, but also because of homogeneity, we can use $V(\mathbf{P}/I,1)$ so only worry about $\mathbf{BH}$ for $\mathbf{P}$ (e.g., $V(P_1, P_2, I)$ would require 4x4 $\mathbf{BH}$, instead use $V(P_1',P_2')$ with 3x3 $\mathbf{BH}$)

<u>Proof</u>:

Trying with $|\mathbf{BH}|$ would be difficult (e.g., $\partial^2V/\partial P_j\partial P_k = -(\partial\lambda/\partial P_k)x_j - \lambda(\partial x_j/\partial P_k)$; both partial derivatives are ambiguous)

Use def'n of quasiconvex: $f(t\mathbf{x}'+(1-t)\mathbf{x}'') \le \max[f(\mathbf{x}'), f(\mathbf{x}'')]$ for $t \in (0,1)$

Let $V(\mathbf{P}^t,I^t) \equiv \text{Max } U(\mathbf{x})$ s.t. $\mathbf{P}^t\cdot\mathbf{x} \le I^t$, were $\mathbf{P}^t = t\mathbf{P}' + (1 - t)\mathbf{P}''$ and $I^t = tI' + (1 - t)I''$

Let $\mathbf{x}^t$ denote the optimal solution to the maximization problem

At optimal solution, budget constraint holds with equality (local nonsatiation in pref)

$\mathbf{P}^t\cdot\mathbf{x}^t = I^t \Rightarrow [t\mathbf{P}' + (1 - t)\mathbf{P}'']\cdot\mathbf{x}^t = tI' + (1 - t)I''$

Budget constraint is linear so we can break it up:

$t[\mathbf{P}'\cdot\mathbf{x}^t - I'] + (1 - t)[\mathbf{P}''\cdot\mathbf{x}^t - I''] = 0$

In order to equal zero, we must have both terms equal zero or one term positive and the other term negative; that means $\mathbf{P}'\cdot\mathbf{x}^t \le I'$ or $\mathbf{P}''\cdot\mathbf{x}^t \le I''$

Consider $\mathbf{P}'\cdot\mathbf{x}^t \le I'$:

That means $\mathbf{x}^t$ is a feasible solution to Max $U(\mathbf{x})$ s.t. $\mathbf{P}'\cdot\mathbf{x}^t \le I'$

The best solution to that problem is $V(\mathbf{P}',I')$

$\therefore\ V(\mathbf{P}',I') \geq U(\mathbf{x}^t) = V(\mathbf{P}^t,I^t)$

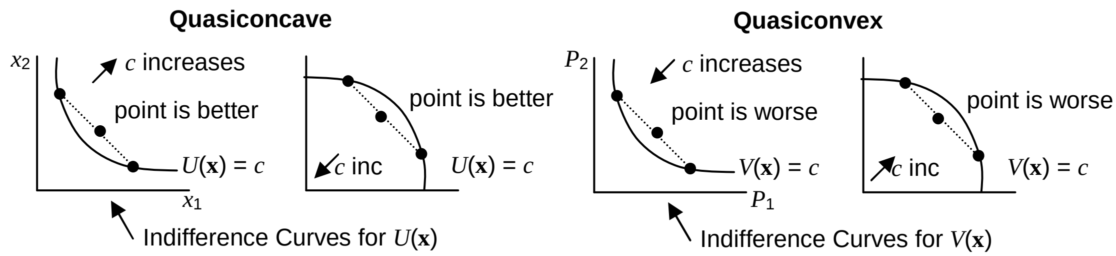
Can use similar argument with $\mathbf{P}''\cdot\mathbf{x}^t \leq I''$ to get $V(\mathbf{P}'',I'') \geq U(\mathbf{x}^t) = V(\mathbf{P}^t,I^t)$

$\therefore\ V(\mathbf{P}^t,I^t) \leq \text{Max}[V(\mathbf{P}',I'),V(\mathbf{P}'',I'')]$, so V is quasiconvex

Other Properties - never used convexity of preferences (or quasiconcave $U(\mathbf{x})$) so there are more

Differentiable - if we assume strictly concave preferences ($U(\mathbf{x})$ strictly quasiconcave), then $V(\mathbf{P},I)$ is continuous and differentiable... hard to prove, but needed for Roy's Identity

Indifference Curves - $V(\mathbf{P},I) = c$ (constant); not same as those in "commodity space" or "x space" which are for regular utility functions (quasiconcave); these are in "price space"; although they're quasiconvex, the direction of improvement is toward the origin (zero prices) not away from it so they have the same shape as the other indifference curves.



Slopes - $U(x_1,x_2) = c$ and $V(P_1,P_2,I) = c$

$$-\left.\frac{dx_1}{dx_2}\right)^{U=c} = \frac{U_1}{U_2} = \frac{P_1}{P_2}; \quad \text{and} \quad -\left.\frac{dP_1}{dP_2}\right)^{V=c} = \frac{V_1}{V_2} = \frac{\partial V/\partial P_1}{\partial V/\partial P_2} = \frac{-\lambda x_1}{-\lambda x_2} = \frac{x_1}{x_2}$$

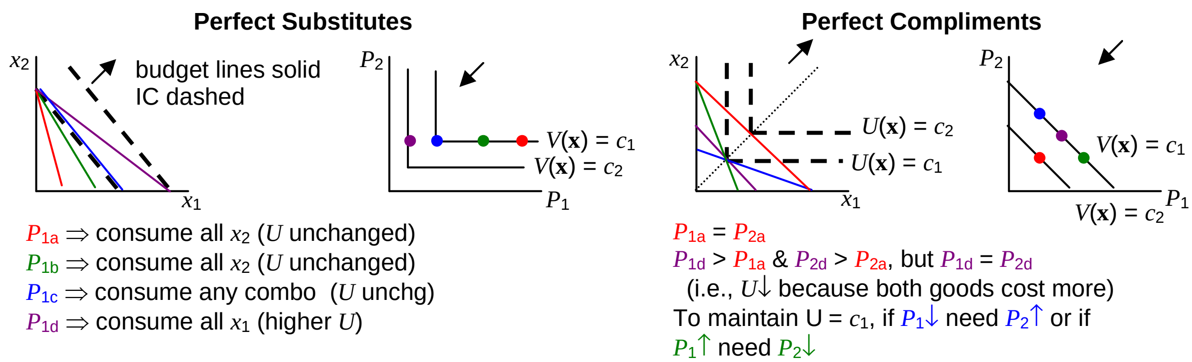
Elasticity of Substitution (σ) -

From utility representation: $\sigma_U = \%\Delta(x_1/x_2)/\%\Delta(P_1/P_2)$ (note: $P_1/P_2 = U_1/U_2 = MRS$)

From indirect utility: $\sigma_V = \%\Delta(P_1/P_2)/\%\Delta(x_1/x_2)$ (note: $x_1/x_2 = V_1/V_2$) $\therefore$ **$\sigma_V = 1/\sigma_U$**

Perfect Substitutes - $\sigma = \infty$; note for a given P_2, starting at very low P_1, we're at a corner solution consuming all good 1; as $P_1\uparrow$, we consume less good 1 so utility declines; eventually we get to point were budget line parallels indifference curve (consume any combination of good 1 and good 2); now as $P_1\uparrow$ we move to other corner (all good 2) and utility remains constant regardless of P_1; creates indifference curve in price space that looks same as perfect compliments in good space (see graph)

Perfect Compliments - $\sigma = 0$; similar argument

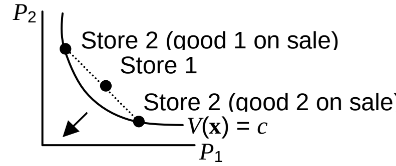


Duality - all information about consumer captured by $U(\mathbf{x})$ is also captured by $V(\mathbf{P},I)$

Bad - no easier to directly estimate; not empirically relevant

Good - theoretically significant for solving certain problems (e.g., taxes on commodities; policies affecting prices, consumer search for lower prices); makes analysis easier

Example - store 1 always sells at same price; store 2 has higher prices, but puts some items on sale randomly so that average prices are same as store 1; which should be preferred? using indirect utility function it's easy to see that store 2 is preferred because it is on a lower (better) indifference curve; consumers benefit more from lower price (stock up on that good) than they're hurt by higher price (just buy less until it goes on sale)



Future Problem - in production theory we'll come to same result for firms (prefer different prices), except reasoning is different; firms prefer to sell more at higher price and less at lower price; inconsistent with consumers; problem will be settled with equilibrium analysis

Example

Covers demand and indirect utility functions

Max $U(x_1,x_2) = x_1x_2$ s.t. $P_1x_1 + P_2x_2 = I$ and x_1 & $x_2 \geq 0$

Quick Observations - know it's = and not ≤ because monotonicity; can ignore corner solutions because $U = 0$ at corner

Lagrangian - $L = x_1x_2 - \lambda(P_1x_1 + P_2x_2 - I)$

K-T Conditions - (1b) $\partial L/\partial x_1 = x_2 - \lambda P_1 = 0$

(2b) $\partial L/\partial x_2 = x_1 - \lambda P_2 = 0$

(3b) $\partial L/\partial \lambda = P_1x_1 + P_2x_2 - I = 0$

Solution - solve (1b) and (2b) for $\lambda = x_2/P_1 = x_1/P_2 \Rightarrow P_1x_1 = P_2x_2$ (spend same amount on both goods)

Now substitutes into (3b): $P_1x_1 + P_2x_2 = I \Rightarrow x_1 = I/2P_1$

Plug back into first eqn: $x_2 = I/2P_2$

Demands:
$$x_1 = \frac{I}{2P_1} \;\&\; x_2 = \frac{I}{2P_2}$$

Check Demand Properties -

1. Defined $\forall$ $\mathbf{P} > \mathbf{0}$ and $I < \infty$... yes
2. "Sort of" continuous... yes (as long as $P_1 > 0$ and $P_2 > 0$)
3. Adding up... $P_1(I/2P_1) + P_2(I/2P_2) = I/2 + I/2 = I$... yes
4. Homogeneous... $x_1(t\mathbf{P},tI) = tI/2tP_1 = I/2P_1 = x_1(\mathbf{P},I)$... yes
5. Convex set... for each I and $\mathbf{P}$, there is a single bundle... yes

Slutsky Equation: (above and beyond)

$$\frac{\partial x_1}{\partial P_1} = S_{11} - x_1\frac{\partial x_1}{\partial I} \Rightarrow S_{11} = \frac{\partial x_1}{\partial P_1} + x_1\frac{\partial x_1}{\partial I} \text{ should be} < 0$$

$$S_{11} = -\frac{I}{2P_1^2} + \left(\frac{I}{2P_1}\right)\frac{1}{2P_1} = -\frac{I}{2P_1^2} + \frac{I}{4P_1^2} = -\frac{I}{4P_1^2} < 0 \text{... yes}$$

$$S_{12} = \frac{\partial x_1}{\partial P_2} + x_2\frac{\partial x_1}{\partial I} = 0 + \left(\frac{I}{2P_2}\right)\frac{1}{2P_1} = \frac{I}{4P_1P_2} > 0 \;\therefore \text{ Hixian substitutes}$$

Indirect Utility Function - plug demands back into $U(x_1,x_2) = x_1x_2$

$V(P_1,P_2,I) = (I/2P_1)(I/2P_2) = I^2/4P_1P_2$

Indirect Utility Function:
$$V(P_1,P_2,I) = \frac{I^2}{4P_1P_2}$$

Check Indirect Utility Function Properties -

1. Defined $\forall$ $\mathbf{P} > \mathbf{0}$ and $I < \infty$... yes
2. Continuous in $\mathbf{P}$ and I... yes
3. Homogeneous... $V(tP_1,tP_2,tI) = t^2I^2/4tP_1tP_2 = I^2/4P_1P_2 = V(P_1,P_2,I)$... yes
4. Increasing in I... $\partial V/\partial I = I/2P_1P_2 > 0$... yes
5. Decreasing in $\mathbf{P}$... $\partial V/\partial P_1 = -I^2/4P_1^2P_2 < 0$; $\partial V/\partial P_2 = -I^2/4P_1P_2^2 < 0$... yes
6. Quasiconvex...

$$|\mathbf{BH}_2| = \begin{vmatrix} 0 & V_1 & V_2 \\ V_1 & V_{11} & V_{12} \\ V_2 & V_{21} & V_{22} \end{vmatrix} = \begin{vmatrix} 0 & \frac{-I^2}{4P_1^2 P_2} & \frac{-I^2}{4P_1 P_2^2} \\ \frac{-I^2}{4P_1^2 P_2} & \frac{I^2}{2P_1^3 P_2} & \frac{I^2}{4P_1^2 P_2^2} \\ \frac{-I^2}{4P_1 P_2^2} & \frac{I^2}{4P_1^2 P_2^2} & \frac{I^2}{2P_1 P_2^3} \end{vmatrix}$$

Simplify by factoring out $I^2/4P_1P_2$ from every row (doesn't change determinant's sign)

$$\begin{vmatrix} 0 & \frac{-1}{P_1} & \frac{-1}{P_2} \\ \frac{-1}{P_1} & \frac{2}{P_1^2} & \frac{1}{P_1 P_2} \\ \frac{-1}{P_2} & \frac{1}{P_1 P_1} & \frac{2}{P_2^2} \end{vmatrix} = \frac{1}{P_1^2 P_2^2} + \frac{1}{P_1^2 P_2^2} - \frac{2}{P_1^2 P_2^2} - \frac{2}{P_1^2 P_2^2} = -\frac{2}{P_1^2 P_2^2} < 0 \text{ ... yes}$$

Roy's Identity: (above and beyond)

$$-\frac{\partial V / \partial P_1}{\partial V / \partial I} = -\frac{-I^2 / 4P_1^2 P_2}{I / 2P_1 P_2} = \frac{I}{2P_1} = x_1 \text{ ... yes}$$

$$-\frac{\partial V / \partial P_2}{\partial V / \partial I} = -\frac{-I^2 / 4P_1 P_2^2}{I / 2P_1 P_2} = \frac{I}{2P_2} = x_2 \text{ ... yes}$$

-Min[-a, -b] = Max[a, b]

Use in definition for quasiconvex (from quasiconcave)

Quasiconcave - $\forall$ $\mathbf{x}'$ and $\mathbf{x}'' \in D$ and $\lambda \in$ (0,1), $f(\lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'') \geq \min[f(\mathbf{x}'), f(\mathbf{x}'')]$

Quasiconvex - $\forall$ $\mathbf{x}'$ and $\mathbf{x}'' \in D$ and $\lambda \in$ (0,1), $f(\lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'') \leq \max[f(\mathbf{x}'), f(\mathbf{x}'')]$

Optimized Value Functions - general rules for optimized value function that is linear in parameters

- **Parameters in Constraint** - quasi-*
- **Parameters in Objective** - not quasi-*
- **Maximization Problem** - *-convex
- **Minimization Problem** - *-concave
- **Example** - $V(\mathbf{P},I)$ is indirect utility function of maximization problem with linear constraint $\mathbf{P}\cdot\mathbf{x} \leq I$; since parameters ($\mathbf{P}$ and I) are in constraint, the indirection $V(\mathbf{P},I)$ is quasiconvex