Incomplete Information

Asymmetric Information - general term for players having different amounts of information
- **Private Information** - player knows something about his own actions that other players don't observe
- **Type of Information** - information could be regarding players' actions, payoffs, or some other intangible aspect of the game (e.g., weather forecast)

Moves of Nature - random events used to incorporate private information

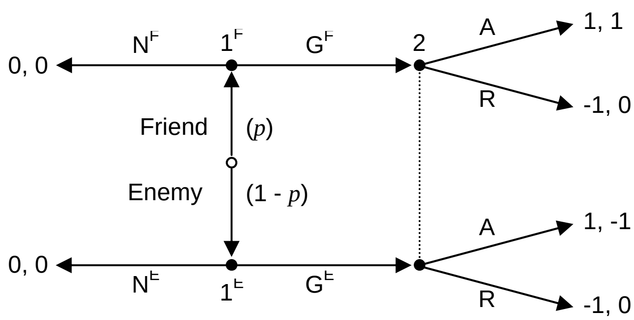
Incomplete Information - refers to games having moves of nature that generate asymmetric information between players; causes problems when trying to maximize (don't have all necessary information)
- **Harsanyi** - paper in Management Science (1967) on how to convert game with incomplete information to game of imperfect, but complete information by inserting random move by nature
  - **Imperfect** - information sets with more than one node (player doesn't know what opponent did or has simultaneous move)
  - **Complete** - everyone knows structure of game
  - **Nature** - "decisions" made according to a fixed probability distribution; no payoff numbers are associated with nature... "decisions" are referred to as **chance nodes** (depicted with open circles)
- **Type** - term used to indicate different moves of nature that a single player privately observes (e.g., different types of home buyers distinguished by different valuations of the house)
  - **Rationality** - requires player who knows his own type to think about what he would have done had he boon another type (i.e., determine best reply for each type)
  - **Bayesian Normal Form** - translates extensive form game with incomplete information into normal form with strategy for each player type and payoffs computed by averaging over random events in game (e.g., if 1 player has 2 types and 2 strategies for each type, there will be $2^2 = 4$ strategies; n types and m strategies means m^n strategies

2 Ways to Solve - each way yields same solution; pick whichever seems easier
- **Incomplete Information** - find best reply for each player type; end up with $(m - 1)n$ equations and $(m - 1)n$ unknowns; unknowns being probability of taking certain strategy given specific player type; note n player types and m strategies so if there are 3 types each with 2 strategies, there are 3 equations (for each player)... see poker example; this technique is not very intuitive and solving the equations is difficult, but could be easier (or less time consuming) than other technique
- **Complete, Imperfect Information** - use m^n strategies in extensive form or Bayesian normal form... ends up with huge matrix; time consuming (but methodical) to enter all the payoffs; after that, it's pretty easy to find pure strategy Nash equilibria (if any exist); finding mixed strategies is much more difficult... not recommended if more than 4x4

Gift Example (book Chpt 24)

Nature first determines the type of player 1 (Friend with probability p or Enemy with probability $1 - p$). Player 1 observes nature's move (knows own type), but player 2 doesn't. Player 1 decides whether to give a gift to player 2, then player 2 decides whether to accept it or not.
- **Bayesian Normal Form** - player 1 has 2 strategies (give & not) and 2 types ∴ 4 strategies total; player 2 has 2 strategies

Payoffs - take expected payoffs based on player 1's type (e.g., for $G^F G^E$ A cell in table: player 1 gets 1 regardless of his type; player 2 gets 1 if player 1 is friend and -1 if enemy ∴ payoff = $1(p) + -1(1 - p) = 2p - 1$



Player 2

Player 1	A	R
$G^F G^E$	1, $2p$ - 1	-1, 0
$G^F N^E$	p, p	-p, 0
$N^F G^E$	1 - p, p - 1	p - 1, 0
$N^F N^E$	0, 0	0, 0

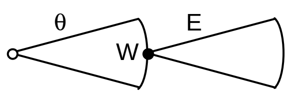
Principal-Agent Problem (book Chpt 25)

Principal-Agent - refers to situation in which one party (the principal) hires another party (the agent) to work on a project on his behalf

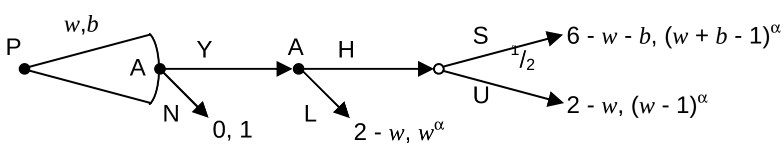
Moral Hazard - stands for setting in which the agent's effort is not verifiable, so the parties cannot write an externally enforced contract specifying a payment as a function of effort

$q(E,\theta)$ - output is function of effort and nature, neither is observed by the principal (owner/manager), but agent (worker) observes θ before deciding E; want to sent up incentives based on E, but also account for θ

Simple Version -

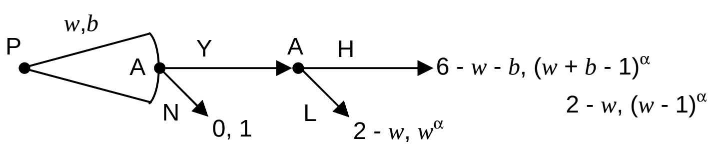


Book Version - Pat is owner and wants to hire Allen; offers contract specifying a wage (w) and bonus (b); Allen then decides whether to accept the contract or not (payoff of 1 through other employment); then Allen decides whether to expend high effort (cost 1) or low effort (no cost); if Allen does high effort, there's a 50-50 chance of the project being successful (but no chance if he does low effort); successful project is worth 6 to Pat and b to Allen; unsuccessful project is worth 2 to Pat and nothing to Allen (but he gets w either way); Allen's utility function is $u_A(x) = x^\alpha$; Pat is risk neutral (usual assumption for principal)



Use backward induction; first get expected payoffs for project

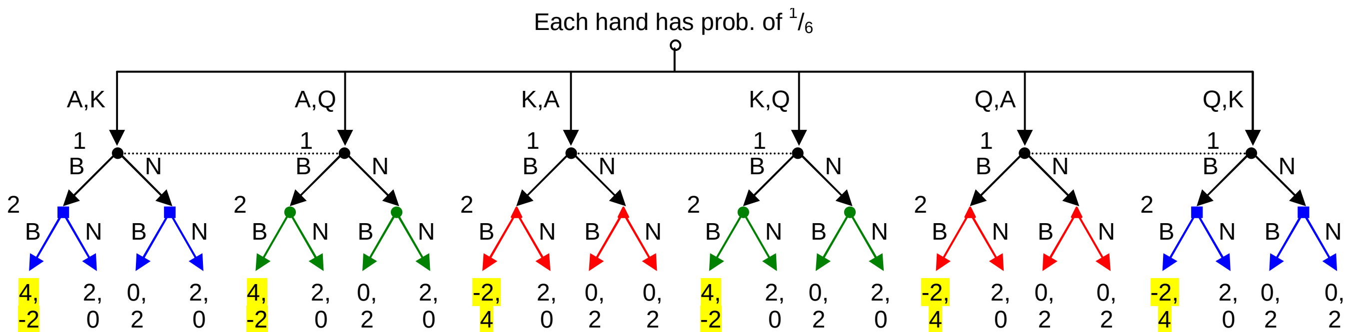
Note: 1/2(6 - w - b) + 1/2(2 - w) = 3 - w/2 - b/2 + 1 - w/2 = 4 - 2 - b/2



Poker Example (from class)

3 card deck: A, K, Q; each player gets 1 card which he sees, but opponents don't see each others' card; each pays \$1 in pot <u>prior to deal</u>; after they see their cards, players simultaneously decide to bet (\$2) or not bet; if both bet or neither bets, player with higher card takes money in pot (with bets, that's \$4 net; \$2 without bets); if only one player bets, he wins the pot automatically (\$2; opponent gets \$0)

Complete, Imperfect Information - can use extensive form (big, but easy) or Bayesian normal form; end up with 3 types of players (A, K, Q) each with 2 strategies (B, N) for a total of 2^3 = 8 strategies for each player; you can see this in the extensive form because each player has 3 information sets (must make a decision in each of them); in each information set, the player only knows his card ∴ form information sets by joining all decision nodes where player knows his card is the same... easy to mark for player 1, but ugly for player 2 (used colors & shapes: red triangles for A, blue squares for K, green circles for Q)



Bayesian Normal Form - have to find expected payoffs for each combination of strategies... 8x8 means 64 of them! (this isn't easy)

BBB and BBB - both players always bet (highlighted payoffs)

Player 1 - 1/6(4 + 4 + -2 + 4 + -2 + -2) = 1

Player 2 - 1/6(-2 + -2 + 4 + -2 + 4 + 4) = 1

Symmetric - payoff in cell ij = reverse payoffs in cell ji

Constant Sum - sum of payoffs = 2 (from \$2 in pot; after that, 1 player wins an additional \$2 only if the other one loses \$2)

Solution - only 1 pure strategy Nash equilibrium (BNN, BNN); that means both players always bet with A, but never bet with K or Q; could be mixed strategies, but they'd be very hard to find this way (have to try all combos of 2, 3, etc.)... try imperfect info technique

Player 2 / **Player 1**

	BBB	BBN	BNB	BNN	NBB	NBN	NNB	NNN
BBB	1, 1	$^1/_3$, $\underline{^5/_3}$	$\underline{^4/_3}$, $^2/_3$	$^2/_3$, $^4/_3$	$\underline{^7/_3}$, $-^1/_3$	$\underline{^5/_3}$, $^1/_3$	$\underline{^8/_3}$, $-^2/_3$	$\underline{2}$, 0
BBN	$\underline{^5/_3}$, $^1/_3$	1, 1	$\underline{^4/_3}$, $^2/_3$	$^2/_3$, $\underline{^4/_3}$	$\underline{^7/_3}$, $-^1/_3$	$\underline{^5/_3}$, $^1/_3$	2, 0	$^4/_3$, $^2/_3$
BNB	$^2/_3$, $\underline{^4/_3}$	$^2/_3$, $\underline{^4/_3}$	1, 1	$\underline{1}$, 1	$^4/_3$, $^2/_3$	$^4/_3$, $^2/_3$	$^5/_3$, $^1/_3$	$^5/_3$, $^1/_3$
BNN	$^4/_3$, $^2/_3$	$\underline{^4/_3}$, $^2/_4$	1, $\underline{1}$	$\underline{1}$, $\underline{1}$	$^4/_3$, $^2/_3$	$^4/_3$, $^2/_3$	1, 1	1, 1
NBB	$-^1/_3$, $\underline{^7/_3}$	$-^1/_3$, $\underline{^7/_3}$	$^2/_3$, $^4/_2$	$^2/_3$, $^4/_3$	1, 1	1, 1	2, 0	$\underline{2}$, 0
NBN	$^1/_3$, $\underline{^5/_3}$	$^1/_3$, $\underline{^5/_3}$	$^2/_3$, $^4/_3$	$^2/_3$, $^4/_3$	1, 1	1, 1	$^4/_3$, $^2/_3$	$^4/_3$, $^2/_3$
NNB	$-^2/_3$, $\underline{^8/_3}$	0, 2	$^1/_3$, $^5/_3$	$\underline{1}$, 1	0, 2	$^2/_3$, $^4/_3$	1, 1	$^5/_3$, $^1/_3$
NNN	0, $\underline{2}$	$^2/_3$, $^4/_3$	$^1/_3$, $^5/_3$	$\underline{1}$, 1	0, $\underline{2}$	$^2/_3$, $^4/_3$	$^1/_3$, $^5/_3$	1, 1

Imperfect Information - consider the card player 1 has and look at the probability of betting in each case

Comments -

1. Payoff tables are same when player 1 has A (because it beats other 2 cards); same happens when player 1 has Q; still need to worry about both payoff tables because we need to worry about opponent's strategy
2. Note for player 1 with an A, B weakly dominates N... since we're interested in finding all Nash equilibria, we can't eliminate weakly dominated strategies because we may eliminate a Nash equilibrium
3. Player 1 knowing his type influences his belief on player 2's type (i.e., 50-50 of having one of remaining 2 cards); that's not always the case
4. Normally we'd repeat this process for player 2, but in this case it's completely symmetrical so we don't have to
5. Symmetry in the game means there will be a symmetric equilibria, but there could be others so we have to do this the long way; we do get to take advantage of symmetry for the computations

Player 1 with an A

Player 2: $^1/_2$ K, $^1/_2$ Q

K:

Player 1	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Q:

Player 1	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Player 1 with a K

Player 2: $^1/_2$ A, $^1/_2$ Q

A:

Player 1	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Q:

Player 1	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Player 1 with an Q

Player 2: $^1/_2$ K, $^1/_2$ Q

K:

Player 1	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Q:

Player 1	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Solving - find best reply function for each player; looking at either choosing B or N (or mixed strategy) based on whether EV^B >, <, or = to EV^N; using π_i and θ_i to represent the probability that player 1 and player 2 (respectively) will bet given they have card i (i = A,K,Q)

Player 1:

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } EV^B - EV^N & = 0 \\ 0 & < 0\text{, where} \end{cases}$$

$EV^B = {}^1/_2(4\theta_K + 2(1 - \theta_K) + 4\theta_Q + 2(1 - \theta_Q)) = \theta_K + \theta_Q + 2$

$EV^N = {}^1/_2(0\theta_K + 2(1 - \theta_K) + 0\theta_Q + 2(1 - \theta_Q)) = 2 - \theta_K - \theta_Q$

$\therefore EV^B - EV^N = \theta_K + \theta_Q$

Repeat for player 1 with a K

$EV^B = {}^1/_2(-2\theta_A + 2(1 - \theta_A) + 4\theta_Q + 2(1 - \theta_Q)) = -2\theta_A + \theta_Q + 2$

$EV^N = {}^1/_2(0\theta_A + 0(1 - \theta_A) + 0\theta_Q + 2(1 - \theta_Q)) = 1 - \theta_Q$

$\therefore EV^B - EV^N = -2\theta_A + 2\theta_Q + 1$

Repeat for player 1 with a Q

$EV^B = {}^1/_2(-2\theta_A + 2(1 - \theta_A) + -2\theta_K + 2(1 - \theta_K)) = -2\theta_A - 2\theta_K + 2$

$EV^N = {}^1/_2(0\theta_A + 0(1 - \theta_A) + 0\theta_K + 0(1 - \theta_K)) = 0$

$\therefore EV^B - EV^N = -2\theta_A - 2\theta_K + 2$

Player 2:

Because of symmetry in the game, player 2's probabilities (θ_A, θ_K, θ_Q) will be exactly the same as the corresponding probabilities for player 1 (i.e., swap θ for π and vice versa for all of player 1's formulas)

Summary: (6 equations and 6 unknowns)

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \theta_K + \theta_Q & = 0 \\ 0 & < 0 \end{cases} \quad (1) \qquad \theta_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \pi_K + \pi_Q & = 0 \\ 0 & < 0 \end{cases} \quad (4)$$

$$\pi_K = \begin{cases} 1 & > \theta_A - \theta_Q \\ [0,1] \text{ as } {}^1/_2 & = \theta_A - \theta_Q \\ 0 & < \theta_A - \theta_Q \end{cases} \quad (2) \qquad \theta_K = \begin{cases} 1 & > \pi_A - \pi_Q \\ [0,1] \text{ as } {}^1/_2 & = \pi_A - \pi_Q \\ 0 & < \pi_A - \pi_Q \end{cases} \quad (5)$$

$$\pi_Q = \begin{cases} 1 & > \theta_A + \theta_K \\ [0,1] \text{ as } 1 & = \theta_A + \theta_K \\ 0 & < \theta_A + \theta_K \end{cases} \quad (3) \qquad \theta_Q = \begin{cases} 1 & > \pi_A + \pi_K \\ [0,1] \text{ as } 1 & = \pi_A + \pi_K \\ 0 & < \pi_A + \pi_K \end{cases} \quad (6)$$

Solving this system requires some creativity to be easy, but it can also be solved if you're very methodical. The basic idea is to start with one of the three possible values for one of the unknowns and then either find values for all the other unknowns that give a solution or find a contradiction (e.g., assume something = 1 then show that it's not). In this case, intuition would suggest always betting with the A... note that you'll never not bet with it (i.e., π_A or $\theta_A = 0$) because then (1) implies $\theta_K + \theta_Q < 0$ and (4) implies $\pi_K + \pi_Q < 0$ which is not possible with probabilities. In order to the theory of always betting with an A, let's assume $\pi_A < 1$ (Note: a more correct notation would be $\pi_A \in (0,1)$)

Assume $\pi_A < 1$

(1) implies $\theta_K + \theta_Q = 0 \Rightarrow \theta_K = \theta_Q = 0$ (because probabilities can't be negative)

Now (5) implies ${}^1/_2 < \pi_A - \pi_Q$ and (6) implies $1 < \pi_A + \pi_K$

From the first inequality we can rule out $\pi_Q = 1$, but that's about it; we're stuck here and have to make another assumption

Because of the symmetry, we'll assume $\theta_A < 1$

(4) implies $\pi_K + \pi_Q = 0 \Rightarrow \pi_K = \pi_Q = 0$

But now look at the implication from (6) above: $1 < \pi_A + \pi_K \Rightarrow$ (sub $\pi_K = 0$) $1 < \pi_A$

That violates our initial assumption that $\pi_A < 1$

$\therefore$ we now know that $\pi_A = 1$ (and by symmetry $\theta_A = 1$)

(1) implies $\theta_K + \theta_Q > 0$ and (2) implies $\pi_K + \pi_Q > 0$

We're stuck again so we need another assumption; look at $\pi_K = 1$

Assume $\pi_K = 1$

(2) implies ${}^1/_2 > \theta_A - \theta_Q = 1 - \theta_Q \Rightarrow \theta_Q > {}^1/_2$

But now (6) implies $1 = \pi_A + \pi_K \Rightarrow 1 = 1 + \pi_K \Rightarrow \pi_K = 0$ which violates our assumption

Assume $\pi_K \in [0,1]$

(2) implies ${}^1/_2 = \theta_A - \theta_Q = 1 - \theta_Q \Rightarrow \theta_Q = {}^1/_2$

But now (6) implies $1 = \pi_A + \pi_K \Rightarrow 1 = 1 + \pi_K \Rightarrow \pi_K = 0$ which violates our assumption

$\therefore$ we now know that $\pi_K = 0$ (and by symmetry $\theta_K = 0$)... if there is a solution

(2) implies ${}^1/_2 < \theta_A - \theta_Q \Rightarrow {}^1/_2 < 1 - \theta_Q \Rightarrow \theta_Q < {}^1/_2$

(6) implies $1 = \pi_A + \pi_K$ which is correct because $\pi_A = 1$ and $\pi_K = 0$

Similarly (using (5) and (3)) we get $\pi_Q < {}^1/_2$

Final solution is:

$\pi_A = \theta_A = 1$
$\pi_K = \theta_K = 0$
$\pi_Q \in [0,{}^1/_2]$ and $\theta_Q \in [0,{}^1/_2]$
That is, both players always bet with an A and never bet with a K; if they have a Q, the can either not bet (that's the pure strategy Nash equilibrium we found earlier) or they can play a mixed strategy were the probability of betting with a Q is anywhere between 0 and ${}^1/_2$ (**Note:** this is bluffing because the Q can't beat either of the other cards; if the players don't bet simultaneously, a player with a Q may be in the hopes that the opponent with a K thinks he has an A and doesn't bet)

Want More?

Can repeat this game with only 2 cards to make it simpler, say player 1 either gets A or Q and player 2 either gets K or J (all with probability ${}^1/_2$) I won't go into all the math, but these some of the steps leading up to the results:

Complete, Imperfect Information - drawing extensive form is simpler than previous version; hardest part is getting the payoffs for the Bayesian strategic form

Player 2

Player 1	BB	BN	NB	NN
BB	$\underline{{}^5/_2}, -{}^1/_2$	${}^3/_2, \underline{{}^1/_2}$	$\underline{3}, -1$	$\underline{2}, 0$
BN	$2, 0$	$\underline{2}, 0$	${}^3/_2, \underline{{}^1/_2}$	${}^3/_2, \underline{{}^1/_2}$
NB	${}^1/_2, \underline{{}^3/_2}$	${}^1/_2, \underline{{}^3/_2}$	$2, 0$	$\underline{2}, 0$
NN	$0, \underline{2}$	$1, 1$	${}^1/_2, {}^3/_2$	${}^3/_2, {}^1/_2$

Results -

- There's no pure strategy Nash equilibrium
- NN is strictly dominated by BB for player 1
- We can now rule out weakly dominated strategies (only concerned with mixed strategies)
- BB weakly dominated by BN and NB weakly dominated by NN for player 2
- Now BB weakly dominates NB for player 1
- Remaining 2x2 game has no pure strategy equilibrium and is symmetric so figure mixed strategy equilibrium is 50-50 for both players
- End result is players mixing of BB and BN for player 1 (i.e. always bet with A, and mix with Q) and BN and NN for player 2 (i.e., never bet with J and mix with K)
- Finding this solution is shown after the incomplete information solution

Incomplete Information - set up payoff tables from each type of player like we did before... note that it's not symmetric anymore; should end up with this:

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \theta_K + \theta_J & = 0 \\ 0 & < 0 \end{cases} \quad (1) \qquad \theta_K = \begin{cases} 1 & > \pi_A - \pi_Q \\ [0,1] \text{ as } {}^1/_2 & = \pi_A - \pi_Q \\ 0 & < \pi_A - \pi_Q \end{cases} \quad (3)$$

$$\pi_Q = \begin{cases} 1 & > \theta_K - \theta_J \\ [0,1] \text{ as } {}^1/_2 & = \theta_K - \theta_J \\ 0 & < \theta_K - \theta_J \end{cases} \quad (2) \qquad \theta_J = \begin{cases} 1 & > \pi_A + \pi_Q \\ [0,1] \text{ as } 1 & = \pi_A + \pi_Q \\ 0 & < \pi_A + \pi_Q \end{cases} \quad (4)$$

Result -

$\pi_A = 1,\ \pi_Q = {}^1/_2,\ \theta_K = {}^1/_2,\ \theta_J = 0$

		Player 2	
		θ	$1 - \theta$
Player 1		**BN**	**NN**
π	**BB**	$^3/_2$, $\underline{^1/_2}$	$\underline{2}$, 0
$1 - \pi$	**BN**	$\underline{2}$, 0	$^3/_2$, $\underline{^1/_2}$

Good Review for Mixed Strategies

This payoff matrix comes from removing the strictly and weakly dominated strategy in the "simple" (2 card) poker game; technically, we still need to worry about the whole thing, but first we're going to solve for π and θ (note that these are the same as π_Q and θ_K in the poker game... look at the strategies and it should make sense)

1. If two (or more) strategies are played with positive probability by a player, then the expected value of those strategies are equal to each other

   Player 1:

   $EV^{BB} = EV^{BN} \Rightarrow {}^3/_2\theta + 2(1 - \theta) = 2\theta + {}^3/_2(1 - \theta) \Rightarrow \theta = {}^1/_2$

   Player 2:

   $EV^{BN} = EV^{NN} \Rightarrow {}^1/_2\pi + 0(1 - \pi) = 0\pi + {}^1/_2(1 - \pi) \Rightarrow \pi = {}^1/_2$

2. The expected payoff of strategies with positive probability must be at least as great as ($\geq$) the expected value from strategies with zero probability

   To check this we have to go back to the original 4x4 payoff matrix... basically we're looking for EV^{NB} and EV^{NN} for player 1 to be < EV^{BB} (or EV^{BN}) and EV^{BB} and EV^{NB} for player 2 to be < EV^{BN} (or EV^{NN})

   Player 1: probabilities for player 2 playing BB, BN, NB, and NN are 0, $^1/_2$, 0, $^1/_2$, respectively

   $EV^{BB} = {}^5/_2(0) + {}^3/_2({}^1/_2) + 3(0) + 2({}^1/_2) = {}^7/_4$

   $EV^{BN} = 2(0) + 2({}^1/_2) + {}^3/_2(0) + {}^3/_2({}^1/_2) = {}^7/_4$... as expected since EVBB = EVBN

   $EV^{NB} = {}^1/_2(0) + {}^1/_2({}^1/_2) + 2(0) + 2({}^1/_2) = {}^5/_4 < {}^7/_4$

   $EV^{NN} = 0(0) + 1({}^1/_2) + {}^1/_2(0) + {}^3/_2({}^1/_2) = {}^5/_4 < {}^7/_4$... works out correctly

   Player 2: probabilities for player 1 playing BB, BN, NB, and NN are $^1/_2$, $^1/_2$, 0, 0, respectively

   $EV^{BB} = -{}^1/_2({}^1/_2) + 0({}^1/_2) + {}^3/_2(0) + 2(0) = -{}^1/_4$

   $EV^{BN} = {}^1/_2({}^1/_2) + 0({}^1/_2) + {}^3/_2(0) + 1(0) = {}^1/_4$

   $EV^{NB} = -1({}^1/_2) + {}^1/_2({}^1/_2) + 0(0) + {}^3/_2(0) = -{}^1/_4$

   $EV^{NN} = 0({}^1/_2) + {}^1/_2({}^1/_2) + 0(0) + {}^1/_2(0) = {}^1/_4$... works out correctly