Proof of Bessel's Relation: Part II
$\frac{d}{dx}(x^{-v}J_v(x)) = -x^{-v}J_{v+1}(x)^*$

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Abstract. Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Bessel functions are therefore especially important for many problems of wave propagation and static potentials. In solving problems in cylindrical coordinate systems, one obtains Bessel functions of integer order ($v = n$); in spherical problems, one obtains half-integer orders ($v = n + \frac{1}{2}$).

Claim: A Bessel function of higher order can be expressed by Bessel functions of lower orders.

$$J_v(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n+v}$$

$$\begin{aligned}
\frac{d}{dx}(x^{-v}J_v(x)) &= \frac{d}{dx}\left\{x^{-v}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n+v}\right\} \\
&= \frac{d}{dx}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!\Gamma(n+v+1)2^{2n+v}}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)x^{2n-1}}{n!\Gamma(n+v+1)2^{2n+v}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n)x^{2n-1}}{n!\Gamma(n+v+1)2^{2n+v}}(-x^{-v})(-x^{v}) \\
&= -x^{-v}\sum_{n=0}^{\infty} \frac{-(-1)^n(2n)x^{2n+v-1}}{n!\Gamma(n+v+1)2^{2n+v}} \\
\text{(for } n=0\text{, the sum is 0)} \qquad &= -x^{-v}\sum_{n=1}^{\infty} \frac{-(-1)^n(2n)x^{2n+v-1}}{n!\Gamma(n+v+1)2^{2n+v}} \\
&= -x^{-v}\sum_{n=1}^{\infty} \frac{-(-1)^n(2)x^{2n+v-1}}{(n-1)!\Gamma(n+v+1)2^{2n+v}} \\
&= -x^{-v}\sum_{n=1}^{\infty} \frac{-(-1)^n x^{2n+v-1}}{(n-1)!\Gamma(n+v+1)2^{2n+v-1}} \\
&= -x^{-v}\sum_{n=1}^{\infty} \frac{-(-1)^n}{(n-1)!\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n+v-1} \\
\text{(let } n=k+1\text{)} \qquad &= -x^{-v}\sum_{k=0}^{\infty} \frac{-(-1)^{k+1}}{k!\Gamma(k+v+2)} \left(\frac{x}{2}\right)^{2k+v+1} \\
[-(-1)^{k+1} = (-1)^k] \qquad &= -x^{-v}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+v+2)} \left(\frac{x}{2}\right)^{2k+v+1} \\
\text{(let } k=n\text{)} \qquad &= -x^{-v}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+2)} \left(\frac{x}{2}\right)^{2n+v+1} \\
&= -x^{-v}J_{v+1}(x) \qquad \square
\end{aligned}$$

*made with LaTeX

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