Proof that the Natural Logarithm Can Be Represented by the Gaussian Hypergeometric Function*

Christopher Nofal†

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Abstract. The Gaussian or ordinary hypergeometric function is a special function represented by the hypergeometric series that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Every second-order linear ODE with three regular singular points can be transformed into this equation.

Claim: $\ln(1+x)$ can be represented by the Gaussian hypergeometric function.

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\begin{aligned}
F(\alpha,\beta,\gamma;x) &:= 1+\sum_{n=1}^{\infty}\frac{\alpha_n\beta_n}{n!\gamma_n}x^n \\
F(1,1,2;-x) &= 1+\sum_{n=1}^{\infty}\frac{1_n 1_n}{n!2_n}(-x)^n \\
\text{(note: } 1_n = n!) \quad &= 1+\sum_{n=1}^{\infty}\frac{n!(-x)^n}{2_n} \\
\text{(note: } 2_n = (n+1)!) \quad &= 1+\sum_{n=1}^{\infty}\frac{(-x)^n}{n+1} \\
&= 1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n+1} \\
xF(1,1,2;-x) &= x\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n+1}\right) \\
&= x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} \\
\text{(let } n = k-1) \quad &= x+\sum_{k=2}^{\infty}\frac{(-1)^{k-1} x^k}{k} \\
[\text{Taylor series expansion for } \ln(1+x)] \quad &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \\
&= \ln(1+x)
\end{aligned}
$$

□

*made with LaTeX

†Christopher Nofal, B.S.C.E., University of Florida College of Engineering.