EAVESDROPPING AND JAMMING COMMUNICATION NETWORKS

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ABSTRACT. Eavesdropping and jamming communication networks is an important part of any military engagement. However, until recently there has been very limited research in the operations research community in this area. Two recent papers by Commander, et al. [5, 6] addressing this problem from an optimization perspective represent the current state of the art. The objective of the problem under consideration is that of determining the optimal number and locations for a set of eavesdropping/jamming devices in order to intercept/supress communication on a wireless network. Here, we survey the problem and highlight the results from the aforementioned papers. Further, we introduce a randomized local search heuristic for the case of jamming a network under complete uncertainty. Our current endeavors are described and directions of further research are addressed.

1. INTRODUCTION

In any military engagement, disrupting the communication mechanism of one's enemy is an important strategic maneuver. Depending on the circumstances the goal may be to intercept the communication, neutralize the communication network, or both. These problems can be modeled in the same manner. Without the loss of generality, we will refer the problem of determining the optimal placement and quantity of eavesdropping or jamming devices to either intercept or disrupt communication on the network as the WIRELESS NETWORK JAMMING PROBLEM (WNJP) and our descriptions will be in this context. Hence jamming devices can also be thought of as eavesdropping devices.

Jamming communication networks is an important problem but has not been intensively researched despite the vast amount of work on optimizing telecommunication systems [26]. Usually, papers studying network interdiction are about preventing jamming and analyzing network vulnerability [8, 25]. To our knowledge, the only literature on network interdiction involving optimal placement of jamming devices is the work of Commander, et al. [6] in which several mathematical programming formulations were given for the deterministic WIRELESS NETWORK JAMMING PROBLEM. In [5], they derive bounds on the optimal number of jamming devices for a subproblem. The only other thoroughly studied cases are problems of minimizing the maximal network flow and maximizing the shortest path between given nodes via arc interdiction using limited resources. Wood [30], Israeli et al. [17], and Cormican et al. [7] studied stochastic and deterministic cases and suggested efficient heuristics. In [23], Smith et al. investigate survivable network designs under several attack scenarios. In another recent paper, Lim and Smith consider an interdiction problem in which the arcs of a multicommodity flow network are disabled in order to minimize the maximum profit obtained by shipping goods on the network [22].

In this paper, we examine the recent work on jammer placement considering both deterministic and uncertain formulations. First, we consider the ideal case in which the topology of the network to be jammed is known. However, since most problems arise in military battlefield scenarios, exact information is often unavailable. In this case, we examine a stochastic framework and use risk measures to evaluate the effectiveness of jamming device placement. Next, we consider the extreme case in which no information about the network

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is assumed other than a bounding area known to contain the network. A randomized local search heuristic for this extreme case is presented and current work on a metaheuristic for a deterministic setup is described. Finally, directions of future research are addressed.

2. ASSUMPTIONS AND DEFINITIONS

In general, the problem of jamming a communication network is to determine the minimum number of jamming devices required to interdict or suppress functionality of the network. Starting with this general statement, more specific ones can be obtained by considering various types of jamming devices and interdiction criteria. Depending on the given information about the communication nodes and the network topology, stochastic or deterministic setups can be constructed [6]. Below we provide assumptions and basic definitions of the considered framework.

We consider radio-transmitting communication networks and jamming devices operating with electromagnetic waves. We assume that the jamming devices have omnidirectional antennas and emit electromagnetic waves in all directions with the same intensity. We also assume that jamming power decreases reciprocally to the squared distance from a device.

Definition 1. *A point (communication node) X is said to be jammed or covered if the cumulative energy received from all jamming devices exceeds some threshold value E:*

$$\sum_i \frac{\lambda}{\mathcal{R}^2(X,i)} \geq E, \tag{1}$$

where $\lambda \in \mathbb{R}$ and $\mathcal{R}(X,i)$ represents the distance from X to jamming device i. This condition can be rewritten as:

$$\sum_i \frac{1}{\mathcal{R}^2(X,i)} \geq \frac{1}{L^2}, \tag{2}$$

where $L = \sqrt{\frac{\lambda}{E}}$.

The latter inequality implies that a jamming device covers any point inside a circle of radius L.

Definition 2. *A connection (arc) between two communication nodes is considered blocked if any of the two nodes is covered.*

3. DETERMINISTIC FORMULATIONS

In this section we derive several formulations for the WNJP first reported by Commander, et al. in [6]. In all of the following problems, we will assume that we have knowledge about the network being jammed. As we will see, despite the assumption that the topology of the network is known entirely, optimal placement of the jamming devices remains computationally difficult.

3.1. Coverage Formulations.

Given a set $\mathcal{M} = \{1, 2, \ldots, m\}$ of communication nodes to be jammed, the goal is to find a set of locations for placing jamming devices in order to suppress the functionality of the network. The cumulative level of jamming energy received at node i is defined as

$$Q_i = \sum_{j=1}^{n} \frac{1}{\mathcal{R}^2(i,j)}, \tag{3}$$

where n is the number of jamming devices. Then, we can formulate the WIRELESS NETWORK JAMMING PROBLEM (WNJP) as the minimization of the number of jamming devices placed, subject to a set of *quality covering* constraints:

$$\textbf{(QCP)} \quad \text{Minimize} \quad n \tag{4}$$

$$\text{s.t.} \quad Q_i \geq C_i, \quad i = 1, 2, \ldots, m. \tag{5}$$

The solution to this problem provides the optimal number of jamming devices needed to ensure a certain jamming threshold C_i is met at every node $i \in \mathcal{M}$. A continuous optimization approach where one is seeking the optimal placement coordinates $(x_j, y_j), j = 1, 2, \ldots, n$, for jamming devices given the coordinates $(X_i, Y_i), i = 1, 2, \ldots, m$, of network nodes, leads to highly non-convex formulations. For example, consider the quality covering constraint for network node i,

$$\sum_{j=1}^{n} \frac{1}{(x_j - X_i)^2 + (y_j - Y_i)^2} \geq C_i.$$

It is easy to verify that this constraint is non-convex. Finding the optimal solution to this nonlinear programming problem would require an extensive amount of computational effort.

To overcome the non-convexity of the above formulation, we propose several integer programming models for the problem. Suppose now that along with the set of communication nodes $\mathcal{M} = \{1, 2, \ldots, m\}$, there is a fixed set $\mathcal{N} = \{1, 2, \ldots, n\}$, of possible locations for the jamming devices. This assumption is reasonable because in real battlefield scenarios, the set of possible placement locations will likely be limited. Define the decision variable x_j as

$$x_j = \begin{cases} 1, & \text{if a jamming device is installed at location } j \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

If we redefine $\mathcal{R}(i, j)$ to be the distance between communication node i and jamming location j, then we have the OPTIMAL NETWORK COVERING (ONC) formulation of the WNJP as

$$\textbf{(ONC)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{7}$$

$$\text{s.t.}$$

$$Q_i x_i \geq C_i, \quad i = 1, 2, \ldots, m \tag{8}$$

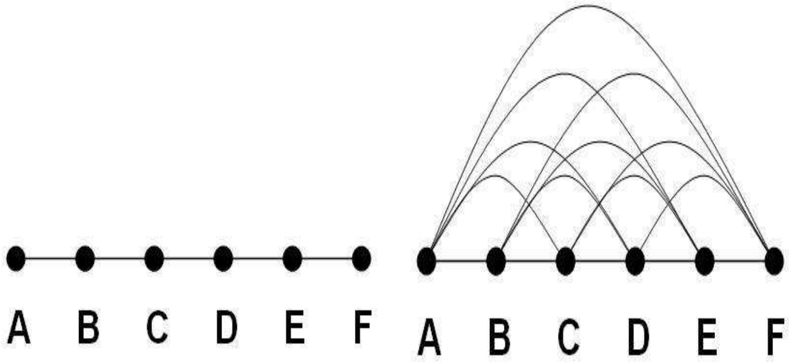
$$x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, n, \tag{9}$$

where C_i is defined as above. Here the objective is to minimize the number of jamming devices used while achieving some minimum level of coverage at each node. The coefficients c_j in (7) represent the costs of installing a jamming device at location j. In a battlefield scenario, placing a jamming device in the direct proximity of a network node may be theoretically possible; however, such a placement might be undesirable due to security considerations. In this case, the location considered would have a higher placement cost than would a safer location. If there are no preferences for device locations, then without the loss of generality,

$$c_j = 1, \quad j = 1, 2, \ldots, n.$$

Though we have removed the non-convex covering constraints, this formulation remains computationally difficult. Notice that ONC is formulated as a MULTIDIMENSIONAL KNAPSACK PROBLEM which is known to be $\mathcal{NP}$-hard in general [11].

3.2. **Connectivity Based Formulations.** In the general WNJP, it is important that the distinction be made that the objective is not simply to jam some of the nodes, but to destroy the functionality of the underlying communication network. In this section, we use tools from graph theory to develop a method for suppressing the network by jamming those nodes with several communication links and derive an alternative formulation of the WNJP.



(a) (b)

Figure 1: (a) Original graph G. (b) Transitive closure of G.

Definition 3. *Given a graph $G = (V, E)$, the* transitive closure *of G is itself a graph $G' = (V, E')$, where $(i, j) \in E'$ if and only if there exists a path from i to j in G. (Figure 1 provides an example of a graph and its transitive closure.)*

Definition 4. *Given a graph $G = (V, E)$, the* connectivity index *of a node $i \in V$ is defined as the number of nodes reachable from that vertex (see Figure 2 for examples).*

To constrain the network connectivity in optimization models, we can impose constraints on the connectivity indices instead of using covering constraints. We assume that the set of communication nodes $\mathcal{M} = \{1, 2, \ldots, m\}$ to be jammed is known and a set of possible locations $\mathcal{N} = \{1, 2, \ldots, n\}$ for the jamming devices is given. Let $d_{ij} = \frac{1}{\mathcal{R}^2(i,j)}$. Furthermore, let $S_i = \sum_{j=1}^{n} d_{ij} x_j$ denote the cumulative level of jamming at node i. Then node i is said to be jammed if S_i exceeds some threshold value C_i. Recall from Definition 2 that communication is severed between nodes i and j if at least one of the nodes is jammed. Further, let $y : V \times V \to \{0, 1\}$ be a surjection where $y_{ij} = 1$ if there exists a path from node i to node j in the jammed network. Lastly, let $z : V \to \{0, 1\}$ where z_i returns 1 if node i is not jammed.

The objective of the CONNECTIVITY INDEX PROBLEM (CIP) formulation of the WNJP is to minimize total jamming cost subject to a constraint that the connectivity index of each node does not exceed some pre-described level L. The corresponding optimization problem is given as:

$$\textbf{(CIP)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{10}$$

s.t.

$$\sum_{j \neq i} y_{ij} \leq L, \; \forall\, i, j \in \mathcal{M} \tag{11}$$

$$M(1 - z_i) \geq S_i - C_i \geq -M z_i, \; \forall\, i \in \mathcal{M} \tag{12}$$

$$x_j \in \{0, 1\}, \; \forall\, j \in \mathcal{N} \tag{13}$$

$$z_i \in \{0, 1\} \; \forall\, i \in \mathcal{M}, \tag{14}$$

$$\forall\, i, j \in \mathcal{M}, y_{ij} = \begin{cases} 1, & \text{if } i \text{ reachable from } j \text{ in the jammed network} \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

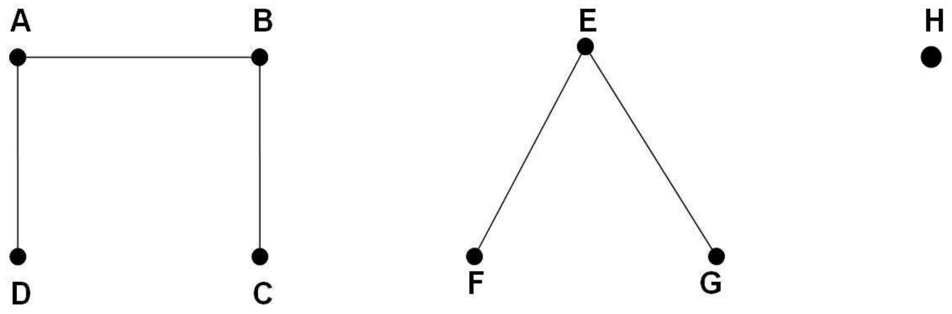


Figure 2: Connectivity Index of nodes A,B,C,D is 3. Connectivity Index of E,F,G is 2. Connectivity Index of H is 0.

where $M \in \mathbb{R}$ is some large constant.

Let $v : V \times V \to \{0,1\}$ and $v' : V \times V \to \{0,1\}$ be defined as follows:

$$v_{ij} \quad = \quad \begin{cases} 1, \text{ if } (i,j) \in E, \\ 0, \text{ otherwise,} \end{cases} \tag{16}$$

and

$$v'_{ij} \quad = \quad \begin{cases} 1, \text{ if } (i,j) \text{ exists in the jammed network,} \\ 0, \text{ otherwise.} \end{cases} \tag{17}$$

With this, we can formulate an equivalent integer program as

$$\textbf{(CIP-1)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j, \tag{18}$$

$$\text{s.t.}$$

$$y_{ij} \geq v'_{ij}, \ \forall\, i,j \in \mathcal{M}, \tag{19}$$

$$y_{ij} \geq y_{ik} y_{kj}, \ k \neq i,j; \ \forall\, i,j \in \mathcal{M}, \tag{20}$$

$$v_{ij}^{'} \geq v_{ij} z_j z_i, \ i \neq j; \ \forall\, i,j \in \mathcal{M}, \tag{21}$$

$$\sum_{j=1}^{m} y_{ij} \leq L, \ j \neq i, \ \forall\, i \in \mathcal{M}, \tag{22}$$

$$M(1 - z_i) \quad \geq \quad S_i - C_i \geq -M z_i, \ \forall\, i \in \mathcal{M}, \tag{23}$$

$$z_i \in \{0,1\}, \ \forall\, i \in \mathcal{M}, \tag{24}$$

$$x_j \in \{0,1\}, \ \forall\, j \in \mathcal{N}, \qquad y_{ij} \in \{0,1\} \ \forall\, i,j \in \mathcal{M}, \tag{25}$$

$$v_{ij} \in \{0,1\}, \ \forall\, i,j \in \mathcal{M}, \qquad v'_{ij} \in \{0,1\}, \ \forall\, i,j \in \mathcal{M}. \tag{26}$$

Lemma 1. *If* CIP *has an optimal solution then,* CIP-1 *has an optimal solution. Further, any optimal solution x^* of the integer programming problem* CIP-1 *is an optimal solution of* CIP.

Proof. It is easy to establish that if i and j are reachable from each other in the jammed network then in CIP-1, $y_{ij} = 1$. Indeed, if i and j are adjacent then there exists a sequence of pairwise adjacent vertices:

$$\{(i_0, i_1), ..., (i_{m-1}, i_m)\}, \tag{27}$$

where $i_0 = i$, and $i_m = j$. Using induction it can be shown that $y_{i_0 i_k} = 1, \ \forall \ k = 1, 2, \ldots, m$. From (19), we have that $y_{i_k i_{k+1}} = 1$. If $y_{i_0 i_k} = 1$, then by (20), $y_{i_0 i_{k+1}} \geq$

$y_{i_0 i_k} y_{i_k i_{k+1}} = 1$, which proves the induction step.

The proven property implies that in CIP-1:

$$\sum_{j \neq i} y_{ij} \geq \text{connectivity index of } i. \tag{28}$$

Therefore, if (x^*, y^*) and (x^{**}, y^{**}) are optimal solutions of CIP-1 and CIP correspondingly, then:

$$V(x^*) \geq V(x^{**}), \tag{29}$$

where V is the objective in CIP-1 and CIP.
As (x^{**}, y^{**}) is feasible in CIP, it can be easily checked that y^{**} satisfies all feasibility constraints in CIP-1 (it follows from the definition of y_{ij} in CIP). So, (x^{**}, y^{**}) is feasible in CIP-1; thus proving the first statement of the lemma.
Hence from CIP-1,

$$V(x^{**}) \geq V(x^*). \tag{30}$$

From (29) and (30):

$$V(x^{**}) = V(x^*). \tag{31}$$

Let us define y such that

$$y_{ij} = 1 \Leftrightarrow j \text{ is reachable from } i \text{ in the network jammed by } x^*.$$

Using (28), (x^*, y) is feasible in CIP-1, and hence optimal. From the construction of y it follows that (x^*, y) is feasible in CIP. Relying on (31) we can claim that x^* is an optimal solution of CIP. The lemma is proved. $\square$

We have therefore established a one-to-one correspondence between formulations CIP and CIP-1. Now, we can linearize the integer program CIP-1 by applying some standard transformations. The resulting linear 0-1 program, CIP-2 is given as

$$\textbf{(CIP-2)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{32}$$

$$\text{s.t.}$$

$$y_{ij} \geq v'_{ij}, \ \forall\, i, j = 1, \ldots, \mathcal{M}, \tag{33}$$

$$y_{ij} \geq y_{ik} + y_{kj} - 1, \ k \neq i, j; \ \forall\, i, j \in \mathcal{M}, \tag{34}$$

$$v'_{ij} \geq v_{ij} + z_j + z_i - 2, \ i \neq j; \ \forall\, i, j \in \mathcal{M}, \tag{35}$$

$$\sum_{j=1}^{I} y_{ij} \leq L, \ j \neq i, \ \forall\, i \in \mathcal{M}, \tag{36}$$

$$M(1 - z_i) \geq S_i - C_i \geq -M z_i, \ \forall\, i \in \mathcal{M}, \tag{37}$$

$$z_i \in \{0, 1\}, \ \forall\, i \in \mathcal{M}, \tag{38}$$

$$x_j \in \{0, 1\}, \ \forall\, j \in \mathcal{N}, \qquad y_{ij} \in \{0, 1\} \, \forall\, i, j \in \mathcal{M}, \tag{39}$$

$$v_{ij} \in \{0, 1\}, \ \forall\, i, j \in \mathcal{M}, \qquad v'_{ij} \in \{0, 1\}, \ \forall\, i, j \in \mathcal{M}. \tag{40}$$

In the following lemma, we provide a proof of equivalence between CIP-1 and CIP-2.

Lemma 2. *If* CIP-1 *has an optimal solution then* CIP-2 *has an optimal solution. Furthermore, any optimal solution* x^* *of* CIP-2 *is an optimal solution of* CIP-1.

Proof. For 0-1 variables the following equivalence holds:

$$y_{ij} \geq y_{ik} y_{kj} \Leftrightarrow y_{ij} \geq y_{ik} + y_{kj} - 1$$

The only differences between CIP-1 and CIP-2 are the constraints:

$$v_{ij}^{'} = v_{ij} z_j z_i \tag{41}$$

$$v_{ij}^{'} \geq v_{ij} + z_i + z_j - 2 \tag{42}$$

Note that (41) implies (42) ($v_{ij} z_j z_i \geq v_{ij} + z_i + z_j - 2$). Therefore, the feasibility region of CIP-2 includes the feasibility region of CIP-1. This proves the first statement of the lemma.

From the last property we can also deduce that for all $x_1,\ x_2$ such that x_1 is an optimal solution of CIP-1, and x_2 is optimal for CIP-2, that

$$V(x_1) \geq V(x_2), \tag{43}$$

where $V(x)$ is the objective of CIP-1 and CIP-2.

Let $(x^*, y^*, v^{'*}, z^*)$ be an optimal solution of CIP-2. Construct $v^{''*}$ using the following rules:

$$v_{ij}^{''*} \quad = \quad \begin{cases} 1, \text{ if } v_{ij} + z_i^* + z_j^* - 2 = 1, \\ 0, \text{ otherwise.} \end{cases} \tag{44}$$

$v_{ij}^{'*} \geq v_{ij}^{''*} \Rightarrow (x^*, y^*, v^{''*}, z^*)$ is feasible in CIP-2 ($y_{ij} \geq v_{ij}^{''*}$), hence optimal (the objective value is $V(x^*)$, which is optimal). Using (44), $(v^{''*}, z^*)$ satisfies:

$$v_{ij}^{''*} = v_{ij} z_j^* z_i^*.$$

Using this we have that $(x^*, y^*, v^{''*}, z^*)$ is feasible for CIP-1. If x_1 is an optimal solution of CIP-1 then:

$$V(x_1) \leq V(x^*) \tag{45}$$

On the other hand, using (43):

$$V(x^*) \leq V(x_1). \tag{46}$$

(45) and (46) together imply $V(x_1) = V(x^*)$. The last equality proves that x^* is an optimal solution of CIP-1. Thus, the lemma is proved. □

We have as a result of the above lemmata the following theorem which states that the optimal solution to the linearized integer program CIP-2 is an optimal solution to the original connectivity index problem CIP.

Theorem 1. *If* CIP *has an optimal solution then* CIP-2 *has an optimal solution. Furthermore, any optimal solution of* CIP-2 *is an optimal solution of* CIP.

Proof. The theorem is an immediate corollary of **Lemma 1** and **Lemma 2**. □

4. DETERMINISTIC SETUP WITH PERCENTILE CONSTRAINTS

As mentioned in Subsection 3.2, to suppress communication on a wireless network does not necessarily imply that all nodes must be jammed. It may be sufficient to jam some percentage of the total number of nodes in order to acquire an effective control over the network. Alternatively, the information assumed about the network may not be entirely accurate. Therefore we formulate the WNJP with percentile constraints which require that some percentage $\alpha \in [0,1]$, of the nodes be jammed. This type of constraint is known as a Value at Risk (VaR) percentile constraint [16].

To incorporate VaR constraints into the ONC and ONC-1 formulations we can easily take advantage of the fact that both formulations are 0-1 programming problems. Let $y : V \rightarrow \{0,1\}$ where

$$y_i \quad = \quad \begin{cases} 1, \text{ if node } i \text{ is covered,} \\ 0, \text{ otherwise.} \end{cases} \tag{47}$$

Then to find the minimum number of locations of jamming devices that will allow for covering $\alpha \cdot 100\%$ of the network nodes with prescribed levels of jamming C_i, we must solve the following integer program

$$\textbf{(ONC-VaR)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{48}$$

$$\text{s.t.}$$

$$\sum_{i=1}^{m} y_i \geq \alpha m, \ i = 1, 2, \ldots, m, \tag{49}$$

$$\sum_{j=1}^{J} d_{ij} x_j \geq C_i y_i, \quad i = 1, 2, \ldots, m, \tag{50}$$

$$x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, n, \tag{51}$$

$$y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, m. \tag{52}$$

Notice that the only difference between this formulation and the ONC formulation is the addition of the m VaR constraints in (49) which ensure that the minimum required percentage of the nodes are jammed. The constraints in (50) enforce the coverage requirement C_i for each node i that is covered.

The approach is quite useful when the network structure is known entirely, because the constraints in ONC-VaR do not guarantee any level of coverage for the nodes with $y_i = 0$. However, this does not make the problem any easier to solve because the VaR type percentile constraints add an additional m integer variables to the problem.

In the same manner, we can reformulate the CONNECTIVITY INDEX PROBLEM formulation to include VaR type constraints. Let $\rho : V \to \mathbb{Z}^+$ be a function such that ρ_i returns the connectivity index of node i. That is, $\rho_i = \sum_{j=1, j \neq i}^{m} y_{ij}$. Further let $w : V \to \{0, 1\}$ be defined as

$$w_i \quad = \quad \begin{cases} 1, \text{ if } \rho_i \leq L, \\ 0, \text{ otherwise.} \end{cases} \tag{53}$$

With this, the connectivity index formulation of WNJP with VaR percentile constraints is given as

$$\textbf{(CIP-VaR)} \quad \text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \tag{54}$$

$$\text{s.t.}$$

$$p_i \leq L w_i + (1 - w_i) M, \ i = 1, 2, \ldots, m, \tag{55}$$

$$\sum_{i=1}^{m} w_i \geq \alpha m, \tag{56}$$

$$x_j \in \{0, 1\}, \ j = 1, 2, \ldots, n \tag{57}$$

$$w_i \in \{0, 1\}, \ i = 1, 2, \ldots, m, \tag{58}$$

$$p_i \in \{0, 1\}, \ i = 1, 2, \ldots, m, \tag{59}$$

where M is some large constant.

As with the ONC-VaR formulation, there are two drawbacks of CIP-VaR. First, there is no control guarantee at all on any of the remaining $(1 - \alpha) \cdot 100\%$ nodes. Secondly, the addition of a m binary variables adds a tremendous computational burden to the problem.

A more tractable approach is to impose a percentile constraint ensuring an average level of coverage $C_{\min}$ for $(1 - \alpha) \cdot 100\%$ of the worst (least) jammed nodes. This type of constraint can be formulated using the concept of Conditional Value-at-Risk (CVaR) [28, 29]. Developed by Rockafellar and Uryasev, CVaR is formally defined as a percentile risk measure constructed for estimation and control of risks in stochastic and uncertain environments. However, CVaR-based optimization techniques can also be applied in a deterministic percentile framework. For a description of CVaR methodology and related optimization techniques, the reader is referred to [28, 29].

Here, we present a formulation of the OPTIMAL NETWORK COVERING problem with CVaR-type percentile constraints resulting in the following mixed integer program:

$$\textbf{(ONC-CVaR)} \quad \text{Minimize} \sum_{j=1}^{n} c_j x_j \tag{60}$$

subject to

$$\zeta + \frac{1}{(1-\alpha)I} \sum_{i=1}^{m} \max\left\{ C_{\min} - \sum_{j=1}^{n} x_j d_{ij} - \zeta,\ 0 \right\} \le 0, \tag{61}$$

$$\zeta \in \mathbb{R}, \tag{62}$$

$$x_j \in \{0,1\}. \tag{63}$$

The CVaR constraint (61) ensures that the average coverage across $(1-\alpha)\cdot 100\%$ of the worst (least) covered nodes exceeds the minimal prescribed level $C_{\min}$. Consequently, the coverage of all other nodes in the network also exceeds $C_{\min}$.

The important point about this formulation is that we have not introduced additional integer variables to the problem in order to add the percentile constraints. Recall, that in ONC-VaR we introduced m discrete variables. Since we have to add only m real variables to replace max-expressions under the summation and a real variable ζ, this formulation is much easier to solve than ONC-VaR. In a similar manner, we can formulate the connectivity index problem with the addition of CVaR constraints as follows:

$$\textbf{(CIP-CVaR)} \quad \text{Minimize} \sum_{j=1}^{n} c_j x_j \tag{64}$$

subject to

$$\zeta + \frac{1}{(1-\alpha)I} \sum_{i=1}^{m} \max\{\rho_i - L - \zeta,\ 0\} \le 0, \tag{65}$$

$$\rho_i \in \mathbb{Z}, \tag{66}$$

$$\zeta \in \mathbb{R}. \tag{67}$$

Recall that ρ_i is the connectivity index of node i. Again, we see that in order to include the CVaR constraint, we only need to add $(m+1)$ real variables to the problem. Computationally, this will be much easier to solve than the CIP-VaR formulation as we will see in a later section.

5. JAMMING UNDER COMPLETE UNCERTAINTY

Since most real-world network jamming situations arise in military battlefield scenarios, exact information about the topology of the adversary's network is oftentimes unknown. Thus, deterministic network interdiction approaches have limited applicability. As we have shown a stochastic approach involving some risk measure for evaluating the efficiency of the jamming device placement may be helpful. However as we previously saw, choosing an appropriate risk measure is a challenging problem in its own right. In this section, we consider an extreme case where there is no a priori information about the topology of the network to be jammed. The only information used in our approach is a bounding area, containing the communication network.

If we ignore the cumulative effect of the jamming devices, then the problem reduces to determining the optimal covering of an area on a plane by circles. This covering problem was solved in 1936 by Kershner [18]. In this section, we will review the recent work of

Commander et al. who have shown that accounting for the cumulative effect of all the devices can lead to significant decreases in costs, i.e. required number of jamming devices [5].

Since we assume no information is known about the network to be jammed, the only reasonable approach is to cover all points in some area known to contain the network. This approach would also be appropriate when some information about the network is available, but is potentially inaccurate.

We consider a case when a communication network is located inside a square. However, all of the following theorems can be formulated for a more general case. For example, to obtain results when the network is contained inside a rectangular region in the plane, the only modification required to the calculations is an appropriate updating of the summation bounds.

An optimal covering is one which contains the minimum number of jamming devices that jam all points in the particular area of interest. However, finding a globally optimal solution for the general problem is difficult [6]. Therefore, we consider a subproblem of covering a square with jamming devices located at the nodes of a uniform grid. The solution to this problem will provide a feasible solution (optimal in certain cases) to the general problem. Suppose the grid step size is R. If the length of a square side a is not a multiple of R, then we cover a bigger square with a side of length $R([\frac{a}{R}]+1)$. See Figure 3 for an example. The optimal solution in the considered problem is a uniform grid with the largest possible step size which covers the square. The problem remains non-trivial, even for this simplified setup.

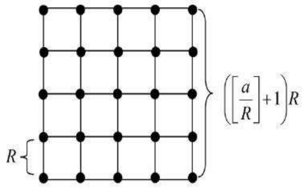


Figure 3: Uniform grid with jamming devices

Lemma 3. *For any covering of a square with a uniform grid, a point which receives the least amount of jamming energy lies inside a corner grid cell (see Figure 4).*

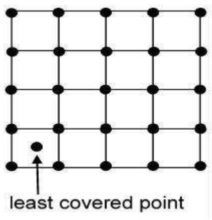


Figure 4: The least covered point is shown in the lower left grid cell.

Proof. Consider a corner cell S_0 and an arbitrary non-corner cell S_i. We prove that for any point $P \in S_i$, there is a corresponding point $P' \in S_0$ such that $E(P) > E(P')$, where $E(X)$ is the cumulative jamming energy from all devices received at point X.

Let P' be a symmetric correspondence of point P inside S_0. Here, symmetry implies that P and P' are equidistant from the sides of their respective cells. We split the square into the four rectangles A, B, C, and D, where A is the rectangle containing cells S_0 and S_i (see Figure 5). Denote the other two corner cells of rectangle A by C_1 and C_2. Let also T_1 and T_2 be points inside C_1 and C_2 respectively, such that T_1PT_2P' is a rectangle with sides parallel to the sides of the square as in Figure 6. Using symmetry we get the following relations:

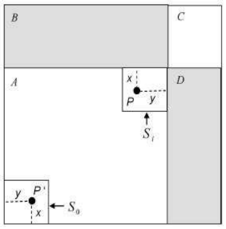


Figure 5: Square Decomposition

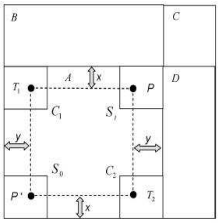


Figure 6: Equivalent Points

$$
\begin{aligned}
E(P',A) &= E(P,A), && (68)\\
E(P',B) < E(T_1,B) &= E(P,B), && (69)\\
E(P',D) < E(T_2,D) &= E(P,D), && (70)\\
E(P',C) &< E(P,C), && (71)
\end{aligned}
$$

where $E(X,I)$ is the cumulative jamming energy from all devices inside rectangle I received by point X. Relations (68) - (71) imply

$$
\begin{aligned}
E(P') &= E(P',A)+E(P',B)+E(P',C)+E(P',D)\\
&< E(P,A)+E(P,B)+E(P,C)+E(P,D) \qquad (72)\\
&= E(P),
\end{aligned}
$$

and the lemma is proved. □

Below we formulate theorems for upper $\overline{R}$ and lower $\underline{R}$ bounds for the optimal grid step size $R^* : \underline{R} < R^* < \overline{R}$. In all formulated theorems, we consider covering a square with side length a.

Theorem 2. *The unique solution of the equation*

$$\frac{1}{2R^2}\Big(\pi\ln(\frac{a}{R}+1)+\pi-3\Big)=\frac{1}{L^2} \qquad (73)$$

is a lower bound $\underline{R}$ *for the optimal grid step size* R^*.

Proof. In Lemma 3, we proved that the least covered point lies inside a corner cell. Consider now a grid with step size R. Without the loss of generality, let $P(x_0, y_0)$ be a point inside the bottom left corner cell as shown in Figure 7. I_1, I_2, and I_3 are cumulative jamming energy received at P by jamming devices located in regions C, A, and B correspondingly. Similarly, I_4 is the jamming energy from the jamming device located at the bottom left node O. With this, the jamming energy received at point P is calculated through the expression

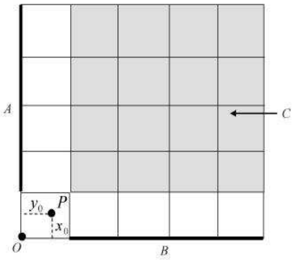


Figure 7: Cumulative emanation of jamming devices.

$$E(P) = I_1 + I_2 + I_3 + I_4, \text{ where} \tag{74}$$

$$I_1 = \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} \frac{1}{(R - x_0 + i \cdot R)^2 + (R - y_0 + j \cdot R)^2}, \tag{75}$$

$$I_2 = \sum_{i=0}^{T-1} \frac{1}{(R - x_0 + i \cdot R)^2 + y_0^2}, \tag{76}$$

$$I_3 = \sum_{j=0}^{T-1} \frac{1}{x_0^2 + (R - y_0 + j \cdot R)^2}, \tag{77}$$

$$I_4 = \frac{1}{x_0^2 + y_0^2}, \tag{78}$$

$$T = \left[\frac{a}{R}\right] + 1. \tag{79}$$

Notice that we can estimate $I_2 + I_3$ as

$$I_2 + I_3 \geq 2 \cdot \sum_{i=0}^{T-1} \frac{1}{R^2(1+i)^2 + R^2} \geq \frac{2}{R^2} \int_0^T \frac{1}{1 + (1+x)^2} dx. \tag{80}$$

This follows from the fact that

$$\sum_{i=0}^{N} f(i) \geq \int_0^{N+1} f(x) dx, \tag{81}$$

where $f(x)$ is a decreasing function. This property can be easily established geometrically. Notice in Figure 8 that the left side of inequality (81) represents the shaded region in the figure, while the right side represents the area under $f(x)$. Continuing from (80) above we

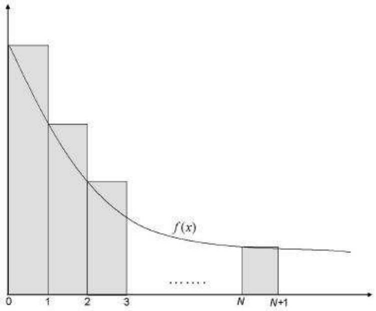


Figure 8: Integral Lower Bound.

have

$$\begin{aligned}\int_0^T \frac{1}{1+(1+x)^2}dx &= \arctan(T+1)-\frac{\pi}{4}\\ &= \frac{\pi}{2}-\arctan\left(\frac{1}{T+1}\right)-\frac{\pi}{4}\\ &\geq \frac{\pi}{4}-\frac{1}{T+1}.\end{aligned} \tag{82}$$

Here and further, we use the inequalities given below:

$$\arctan(x) \leq x,\ 0 \leq x \leq 1, \tag{83}$$

$$\arctan(x) \geq x-\frac{x^3}{3}, 0 \leq x \leq 1. \tag{84}$$

Now combining (80) and (82), we obtain

$$I_2+I_3 \geq \frac{2}{R^2}\left(\frac{\pi}{4}-\frac{1}{T+1}\right). \tag{85}$$

We also have the following approximation for I_4 which follows clearly

$$I_4 \geq \frac{1}{2R^2}. \tag{86}$$

For estimating I_1 we use a property similar to (81), but in a higher dimension. Namely,

$$\sum_{i=0}^{N}\sum_{j=0}^{N} f(i,j) \geq \int_0^{N+1}\int_0^{N+1} f(x,y)dxdy, \tag{87}$$

where as above, $f(x,y)$ is a decreasing function of x and y. Using this inequality, we derive the following approximation for I_1.

$$\begin{aligned}I_1 &\geq \int_0^T\int_0^T \frac{dxdy}{(R-x_0+x\cdot R)^2+(R-y_0+y\cdot R)^2}\\ &\geq \int_0^T\int_0^T \frac{dxdy}{(R+x\cdot R)^2+(R+y\cdot R)^2}\\ &= \frac{1}{R^2}\int_1^{T+1}\int_1^{T+1}\frac{dxdy}{x^2+y^2}.\end{aligned} \tag{88}$$

Furthermore,

$$\begin{aligned}\frac{1}{R^2}\int_1^{T+1}\int_1^{T+1}\frac{dxdy}{x^2+y^2} &= \int_1^{T+1}\frac{1}{x}\arctan\left(\frac{T+1}{x}\right)dx-\int_1^{T+1}\frac{1}{x}\arctan\left(\frac{1}{x}\right)dx\\ &\geq \int_1^{T+1}\frac{1}{x}\arctan\left(\frac{T+1}{x}\right)dx-\int_1^{T+1}\frac{dx}{x^2}\\ &= \int_1^{T+1}\frac{1}{x}\left(\frac{\pi}{x}-\arctan\left(\frac{x}{T+1}\right)\right)dx-1+\frac{1}{T+1} \qquad (89)\\ &= \frac{\pi}{2}\ln(T+1)-1+\frac{1}{T+1}-\int_0^{T+1}\frac{1}{x}\arctan\left(\frac{x}{T+1}\right)dx\\ &\geq \frac{\pi}{2}\ln(T+1)-1+\frac{1}{T+1}-\int_0^{T+1}\frac{1}{x}\left(\frac{x}{T+1}\right)dx\\ &= \frac{\pi}{2}\ln(T+1)-2\left(1-\frac{1}{T+1}\right).\end{aligned}$$

Combining this result with (88) we have

$$I_1 \geq \frac{1}{R^2}\left(\frac{\pi}{2}\ln(T+1)-2\left(1-\frac{1}{T+1}\right)\right). \qquad (90)$$

Summing (85), (86), and (90) we obtain an overestimate of the total coverage at point P. That is

$$\begin{aligned}E(P) &\geq \frac{1}{R^2}\cdot\left(\frac{\pi}{2}\ln(T+1)-2+\frac{2}{T+1}+\frac{\pi}{2}-\frac{2}{T+1}+\frac{1}{2}\right)\\ &= \frac{1}{R^2}\left(\frac{\pi}{2}\ln(T+1)+\frac{\pi}{2}-\frac{3}{2}\right) \qquad (91)\\ &\geq \frac{1}{2R^2}\left(\pi\cdot\ln\left(\frac{a}{R}+1\right)+\pi-3\right).\end{aligned}$$

To guarantee coverage of point P, it is sufficient to claim that

$$f(R)=\frac{1}{2R^2}\left(\pi\cdot\ln\left(\frac{a}{R}+1\right)+\pi-3\right)\geq\frac{1}{L^2}. \qquad (92)$$

Since $f(R)$ is monotonically decreasing on $(0,+\infty)$, the largest R satisfying the above inequality is the unique solution $\underline{R}$ of the equation

$$f(R)=\frac{1}{L^2}. \qquad (93)$$

Thus, a uniform grid with step size $\underline{R}$ jams any point P inside a corner cell. According to Lemma 3, the grid jams the least covered point in the square implying that the whole square is jammed. Thus we have the desired result. $\square$

Since the function $f(R)=\frac{1}{2R^2}(\pi\ln(\frac{a}{R}+1)+\pi-3)$ is monotonic, equation (73) can be easily solved using a numerical procedure such as a binary search [20]. Therefore, using (73), we can obtain a step size $\underline{R}$ such that the corresponding uniform grid covers the entire square. Further, the number of jamming devices in the grid does not exceed

$$N_1=\left(\frac{a}{\underline{R}}+2\right)^2. \qquad (94)$$

A more straightforward solution of the initial problem could be based on the property that a jamming device covers all the points inside a circle of radius L as mentioned in Definition 1. Using that, we could reduce the problem to finding the optimal covering of a square with circles of radius L. A direct result from [18] (that was mentioned in [25]) is that in the limit, the minimum number of circles to cover an area a^2 is

$$N_2 = \frac{2a^2}{3\sqrt{3L^2}}. \tag{95}$$

To compare the approaches, we consider the ratio

$$\begin{aligned} \frac{N_2}{N_1} &= \left(\frac{R}{L^2}\right)\frac{2}{3\sqrt{3}}\frac{1}{(1+2\frac{R}{a})^2} \\ &= \frac{2x^2}{3\sqrt{3}}\frac{1}{(1+\frac{2x}{k})^2}, \end{aligned} \tag{96}$$

where $x = \frac{R}{L}$ and $k = \frac{a}{L}$. Using these substitutions, equation (73) can be rewritten in terms of variables x and k as follows

$$\frac{1}{x^2}\left(\pi \ln\left(\frac{k}{x}+1\right)+\pi-3\right) = 2. \tag{97}$$

By solving (97) for different values of k, one can find corresponding values of x and $\frac{N_2}{N_1}$. To evaluate the advantage of the uniform grid approach over the naive one, we provide some computational results in the Table 1. From the table, we see that as k increases, the advantage of using our approach becomes more significant. In fact, it can be proved that $\lim_{a\to\infty}\frac{N_2}{N_1} = \infty$. This will follow as a corollary of Theorem 4.

k	x	$\frac{N_2}{N_1}$
10^2	2.44	2.3
10^4	3.54	4.8
10^6	4.40	7.5
10^8	5.14	10.2

Table 1: Comparing $\frac{N_2}{N_1}$ for various values of k.

To establish the quality of the lower bound rigorously, we need to first establish a similar result for an upper bound. This follows in the next theorem.

Theorem 3. *The unique solution of the equation*

$$\frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right)-\frac{1}{6(\frac{a}{R}+1)}+\frac{\pi}{2}+\frac{19}{3}\right) = \frac{1}{L^2} \tag{98}$$

is an upper bound $\overline{R}$ of the optimal grid step size R^*.

Proof. Let $P(x_0, y_0)$ be the least jammed point, that lies inside a corner cell according to Lemma 3. Without the loss of generality, as in the proof of Theorem 2, we assume that P is inside the bottom left corner cell. The jamming energy received at point P is calculated through the expressions (74) - (79). Since P is the least covered point, the following inequality holds.

$$E(P) \leq E\left(P'\left(x=\frac{R}{2}, y=0\right)\right) = I_1' + I_2' + I_3' + I_4', \text{ where} \tag{99}$$

$$I_1' = \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} \frac{1}{(\frac{R}{2}+i\cdot R)^2 + (R+j\cdot R)^2}, \tag{100}$$

$$I_2' = \sum_{i=0}^{T-1} \frac{1}{(\frac{R}{2}+i\cdot R)^2}, \tag{101}$$

$$I_3' = \sum_{j=0}^{T-1} \frac{1}{(\frac{R}{2})^2 + (R+j\cdot R)^2}, \tag{102}$$

$$I_4' = \frac{1}{(\frac{R}{2})^2}. \tag{103}$$

I_2' and I_3' can be estimated through integrals similarly to the techniques used in the proof of Theorem 2. The following inequality holds

$$\sum_{i=1}^{N} f(i) \leq \int_0^N f(x)dx, \tag{104}$$

where $f(x)$ is a decreasing function. This property can also be proven geometrically. Figure 9 represents a graphical interpretation of this relation. The left side of the inequality is represented by the shaded area. The right side of (104) is the area under $f(x)$. With this property we have from (101) that

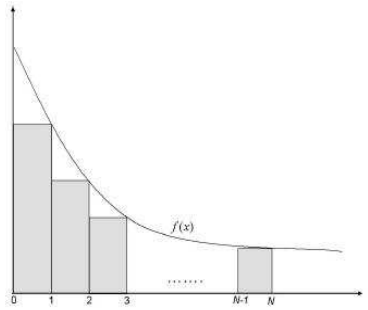


Figure 9: Integral Upper Bound.

$$\begin{aligned} I_2' &\leq \frac{1}{(\frac{R}{2})^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2}+x\cdot R)^2} \\ &= \frac{1}{R^2}\left(6 - \frac{1}{T-\frac{1}{2}}\right). \end{aligned} \tag{105}$$

Furthermore, using inequalities (83) and (84), we see that (102) is estimated by

$$\begin{aligned} I_3' &\leq \frac{1}{(\frac{R}{2})^2 + (R + x \cdot R)^2} \\ &= \frac{2}{3R^2} + \frac{2}{R^2}\left(\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{2T}\right)\right) \\ &\leq \frac{2}{3R^2} + \frac{2}{R^2}\left(\frac{1}{2} - \frac{1}{2T} + \frac{1}{24T^3}\right) \qquad (106) \\ &= \frac{1}{R^2}\left(\frac{5}{3} - \frac{1}{T} + \frac{1}{12T^3}\right). \end{aligned}$$

To estimate I_1' a property similar to (104) can be used. This inequality is given by

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f(i,j) \leq \int_0^N \int_0^N f(x,y)dxdy + \int_0^N f(x,0)dx + \int_0^N f(0,y)dy, \qquad (107)$$

where $f(x,y)$ is a decreasing function of x and y. With the above inequality,

$$\begin{aligned} I_1' &\leq \frac{1}{(\frac{R}{2}^2) + R^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2})^2 + (R + x \cdot R)^2} + \int_0^{T-1} \frac{dx}{(\frac{R}{2} + x \cdot R)^2 + R^2} + \\ &+ \int_0^{T-1}\int_0^{T-1} \frac{dxdy}{(\frac{R}{2} + x \cdot R)^2 + ((R + y \cdot R)^2} \\ &= \frac{4}{5R^2} + \frac{C}{R^2} + \frac{1}{R^2}\int_0^{T-1}\int_0^{T-1} \frac{d(x+\frac{1}{2})dy}{(\frac{1}{2}+x)^2 + (y+1)^2}, \text{ where} \qquad (108) \end{aligned}$$

$$\begin{aligned} C &= 2\arctan(2T) - \arctan(2) + \arctan\left(T - \frac{1}{2}\right) - \frac{\pi}{2} \\ &= \frac{\pi}{2} - 2\arctan\left(\frac{1}{2T}\right) + \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{2}{2T-1}\right) \qquad (109) \\ &\leq \frac{\pi}{2} - 2\left(\frac{1}{2T} - \frac{1}{24T^3}\right) + \frac{1}{2} - \left(\frac{2}{2T-1} - \frac{8}{3(2T-1)^3}\right) \\ &\leq \frac{\pi+1}{2}. \end{aligned}$$

The double integral in (108) is bounded as follows

$$\begin{aligned} &\int_0^{T-1}\int_0^{T-1} \frac{d(x+\frac{1}{2})dy}{(\frac{1}{2}+x)^2 + (y+1)^2} = \int_{\frac{1}{2}}^{T-\frac{1}{2}}\int_1^T \frac{dtdy}{t^2+y^2} \\ &= \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\arctan\left(\frac{T}{t}\right) - \arctan\left(\frac{1}{t}\right)\right) dt \\ &\leq \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{\pi}{2} - \arctan\left(\frac{t}{T}\right)\right) dt - \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{1}{t} - \frac{1}{3t^3}\right) dt \qquad (110) \\ &\leq \frac{\pi}{2}\left(\ln\left(T - \frac{1}{2}\right) - \ln\left(\frac{1}{2}\right)\right) - \int_{\frac{1}{2}}^{T-\frac{1}{2}} \frac{1}{t}\left(\frac{t}{T} - \frac{t^3}{3T^3}\right) dt - \\ &- \left(\frac{4}{3} - \frac{1}{T-\frac{1}{2}} + \frac{1}{6(T-\frac{1}{2})^2}\right) \\ &= \frac{\pi}{2}\ln(2T-1) - \frac{20}{3} + \frac{5}{6T} + \frac{1}{12T^2} - \frac{1}{36T^3} + \frac{1}{T-\frac{1}{2}} - \frac{1}{6(T-\frac{1}{2})^2} \\ &< \frac{\pi}{2}\ln(2T-1) - \frac{20}{3} + \frac{5}{6T} + \frac{1}{T-\frac{1}{2}} - \frac{1}{12(T-\frac{1}{2})^2}. \end{aligned}$$

Combining the results from (108), (109), and (110) gives the overestimate for I_1' as

$$I_1' < \frac{1}{R^2}\left(\frac{\pi}{2}\ln(2T-1)+\frac{\pi}{2}-\frac{16}{3}+\frac{5}{6T}+\frac{1}{T-\frac{1}{2}}-\frac{1}{12(T-\frac{1}{2})^2}\right). \qquad (111)$$

Recall equation (99) stated $E(P) \leq I_1' + I_2' + I_3' + I_4$. So using the expression for I_4' given in (103) and the overestimates for I_1', I_2', and I_3' derived in equations (111), (105), and (106) respectively, we obtain

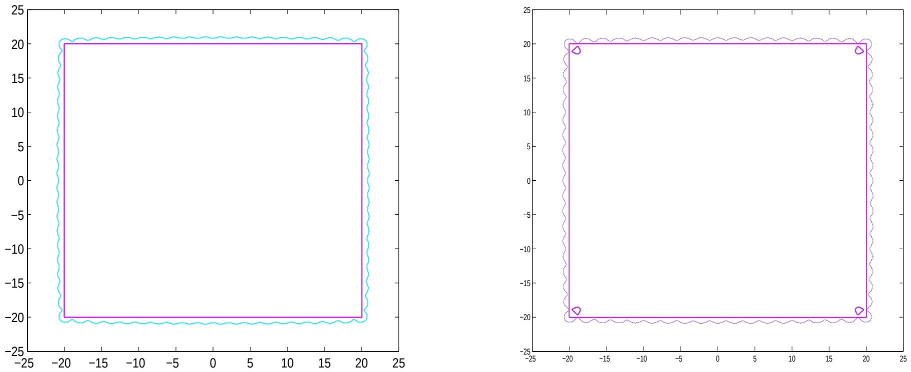
$$E(P) \quad \leq \quad \frac{1}{R^2}\left(\frac{\pi}{2}\ln(2T-1)-\frac{1}{6T}+\frac{\pi}{2}+\frac{19}{3}\right). \qquad (112)$$

Finally, if we let $T = [\frac{a}{R}]+1 \leq \frac{a}{R}+1$, we get

$$E(P) < \frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right)-\frac{1}{6(\frac{a}{R}+1)}+\frac{\pi}{2}+\frac{19}{3}\right) \qquad (113)$$

The function $f(R) = \frac{1}{R^2}\left(\frac{\pi}{2}\ln\left(\frac{2a}{R}+1\right)-\frac{1}{6(\frac{a}{R}+1)}+\frac{\pi}{2}+\frac{19}{3}\right)$ is monotone, hence the equation $f(R) = \frac{1}{L^2}$ has a unique solution $\overline{R}$. Equation (113) implies that a grid with step size $\overline{R}$ does not cover the entire square. That is, there exists at least one point P that remains uncovered. Thus $\overline{R}$ is an upper bound for the optimal grid covering problem. Since the optimal grid step size $R^* < \overline{R}$, the theorem is proved. $\square$

In Figure 5, we see an example in which we are covering at 40×40 square and the required jamming level at each point is 3.0 units. In part (a), we see the coverage associated with the required number of devices from the lower bound of Theorem 2. In this case, $20^2 = 400$ jamming devices are used to cover the area. Notice that there are no holes in the region. This, together with the scallop shell outside the bounding box indicates that all points within the region are covered. In part (b), we see the coverage corresponding to the placement of the jamming devices on a uniform grid according to the upper bound of Theorem 3. Here, the required number of devices is $19^2 = 361$. Notice the holes located at the four corners of the region indicating that these points are uncovered. This validates the theoretical results obtained in Theorem 2 and Theorem 3.



(a) (b)

Figure 10: (a) The coverage of when jamming devices are placed according to the lower bound from Theorem 2. The total number of jamming devices required is $20^2 = 400$. (b) We see the coverage associated with the result obtained from Theorem 3. In this case, $19^2 = 361$ devices are placed. Notice the corner points are not jammed.

Now that we have established both upper and lower bounds for an optimal grid step size, we can determine the quality of the bounds. The result is obtained in the following theorem.

Theorem 4.

$$\lim_{a\to\infty} \frac{\overline{R}}{\underline{R}} = 1, \tag{114}$$

where $\overline{R}$ and $\underline{R}$ are bounds obtained from equations (73) and (98), correspondingly. Moreover, the following inequality holds:

$$1 \le \frac{\overline{R}}{\underline{R}} \le \sqrt{1 + \frac{c}{\ln(a)}}, \tag{115}$$

for constants $M \in \mathbb{R}, c \in \mathbb{R}$, such that $\overline{R} > M$.

Proof. By letting $x = \frac{\underline{R}}{L}$ and $y = \frac{\overline{R}}{L}$, equations (73) and (98) can be respectively rewritten as

$$a = L \cdot x \left(e^{\frac{2}{\pi}(x^2+\frac{3}{2})-1} - 1\right) \text{, and} \tag{116}$$

$$\frac{\pi}{2} \ln\left(\frac{2a}{L \cdot y} + 1\right) = y^2 - \frac{19}{3} - \frac{\pi}{2} + \frac{L \cdot y}{6(a + L \cdot y)}. \tag{117}$$

To prove the theorem, we need to show that

$$\lim_{a\to\infty} \frac{y}{x} = 1, \tag{118}$$

where $x > 0$ and $y > 0$ are solutions of (116) and (117), correspondingly. From (117), we obtain

$$\frac{\pi}{2} \ln\left(\frac{2a}{L \cdot y} + 1\right) > y^2 - C_1, \text{ where} \tag{119}$$

$$C_1 = \frac{19}{3} + \frac{\pi}{2}, \text{ and} \tag{120}$$

$$a > \frac{L \cdot y}{2}\left(e^{\frac{2}{\pi}(y^2 - C_1)} - 1\right). \tag{121}$$

From (116) and (121) we see that

$$x\left(e^{\frac{2}{\pi}(x^2+C_2)} \cdot C_3 - 1\right) > \frac{y}{2}\left(e^{\frac{2}{\pi}(y^2-C_1)} - 1\right), \text{ where} \tag{122}$$

$$C_2 = \frac{3}{2}, \text{and} \tag{123}$$

$$C_3 = e^{-1}. \tag{124}$$

Since $y \cdot L$ and $x \cdot L$ are upper and lower bounds, correspondingly, the following relation holds

$$\frac{y}{x} > 1. \tag{125}$$

With (116) and (125) above, we can also conclude that

$$\lim_{a\to\infty} x = \infty \quad \text{and} \quad \lim_{a\to\infty} y = \infty. \tag{126}$$

For all $M \in \mathbb{R}$, where $M > \sqrt{C_1}$, there exists $Q \in \mathbb{R}$ such that (122) can be reduced to

$$\frac{y}{x} \quad < \quad Q \cdot e^{\frac{2}{\pi}(x^2 - y^2)}, \text{ and } y > M. \tag{127}$$

Moreover, for $c = \frac{\pi}{2}\ln(Q)$ the following inequality holds

$$\left(\frac{y}{x}\right)^2 - 1 \le \frac{c}{x^2}, \text{ and } y > M. \tag{128}$$

Assume for the sake of contradiction that the inequality in (128) does not hold for some (x^*, y^*). That is assume that $\left(\frac{y^*}{x^*}\right)^2 - 1 > \frac{c}{x^{*2}}$. Using (127) we have

$$\frac{y^*}{x^*} < Q \cdot e^{-\frac{2}{\pi}x^{*2}\left((\frac{y^*}{x^*})^2 - 1\right)} < Q \cdot e^{-\frac{2}{\pi}x^{*2}\cdot\frac{c}{x^{*2}}} = 1, \tag{129}$$

which contradicts (125).

Applying (125) and (128) we get

$$1 < \frac{y}{x} \leq \sqrt{1 + \frac{c}{x^2}}, \text{ and } y > M. \tag{130}$$

Letting a tend to ∞ and taking (126) into account, we see that in fact

$$\lim_{a \to \infty} \frac{y}{x} = 1. \tag{131}$$

Finally, by using (130) and (116), the following relation can be obtained

$$1 < \frac{y}{x} \leq \sqrt{1 + \frac{k}{\ln(a)}}, \tag{132}$$

for some constant $k \in \mathbb{R}$, when $y > M$. Thus, the theorem is proved. □

6. CASE STUDIES AND ALGORITHMS

In order to demonstrate the advantages and disadvantages of the proposed formulations for the WNJP, we present several case studies and algorithms. First, we describe two case studies comparing the optimal solutions for the deterministic formulations. Then we describe a randomized local search algorithm for the case of jamming under complete uncertainty and present some preliminary results [4].

6.1. Case Studies for Deterministic Formulations.

The experiments described in this subsection were performed on a PC equipped with a 1.4MHz Intel Pentium® 4 processor with 1GB of RAM, working under the Microsoft Windows® XP SP1 operating system. In the first study, an example network is given and the problem is modeled using the proposed coverage formulation. The problem is then solved exactly using the commercial integer programming software package, CPLEX®. Next, we modify the problem to include VaR and CVaR constraints and again use CPLEX® to solve the resulting problems. Numerical results are presented and the three formulations are compared. In the second case study, we model and solve the problem using the connectivity index formulation. We then include percentile constraints re-optimize. Finally, we analyze the results.

Optimal Solutions	Regular Constraints	VaR Constraints
Number of Jammers	6	4
Level of Jamming	100% $\forall$ nodes	100% for 96% of nodes, 85% (of reqd.) for 4% of nodes
CPLEX® Time	0.81 sec	0.98 sec

Table 2: Optimal solutions using the coverage formulation with regular and VaR constraints.

6.1.1. *Coverage Formulation.*

Here we present two networks and solve the WNJP using the network covering (ONC) formulation. The first network has 100 communication nodes and the number of available jamming devices is 36. The cost of placing a jamming device at location j, c_j is equal to 1 for all locations. This problem was solved using the regular constraints and the VaR type constraints. Recall that there is a set of possible locations

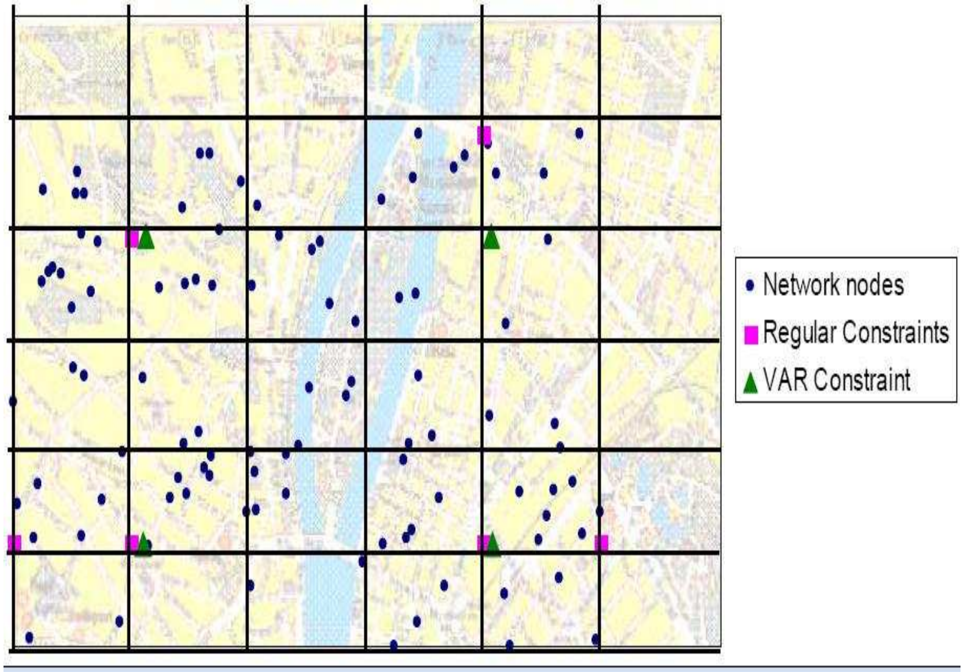


Figure 11: Case study 1. The placement of jamming devices is shown when the problem is solved using the original and VaR constraints.

at which jamming devices can be placed. In these examples, this set of points constitutes a uniform grid over the battlespace. The placement of the jamming devices from each solution can be seen in Figure 11. The numerical results detailing the level of jamming for the network nodes is given in Table 2. Notice that the VaR solution called for 33% less jamming devices than the original problem while providing almost the same jamming quality.

Opt Solns	Reg (all)	VaR (.9 conf)	CVaR (.7 conf)
# Jammers	9	8	7
Jamming Level	100% $\forall$ nodes	100% for 90% of nodes, 72% for 10% of nodes	100% for 57% of nodes, 90% for 20% of nodes, 76% for 23% of nodes
CPLEX® Time	15 sec	15h 55min 11sec	41 sec

Table 3: Optimal solutions using the coverage formulation with regular and VaR, and CVaR constraints.

In the second example, the network has 100 communication nodes and 72 available jammers. This problem was solved using the regular constraints as well as both types of percentile constraints. The resulting graph is shown in Figure 12. The corresponding numerical results are given in Table 3.

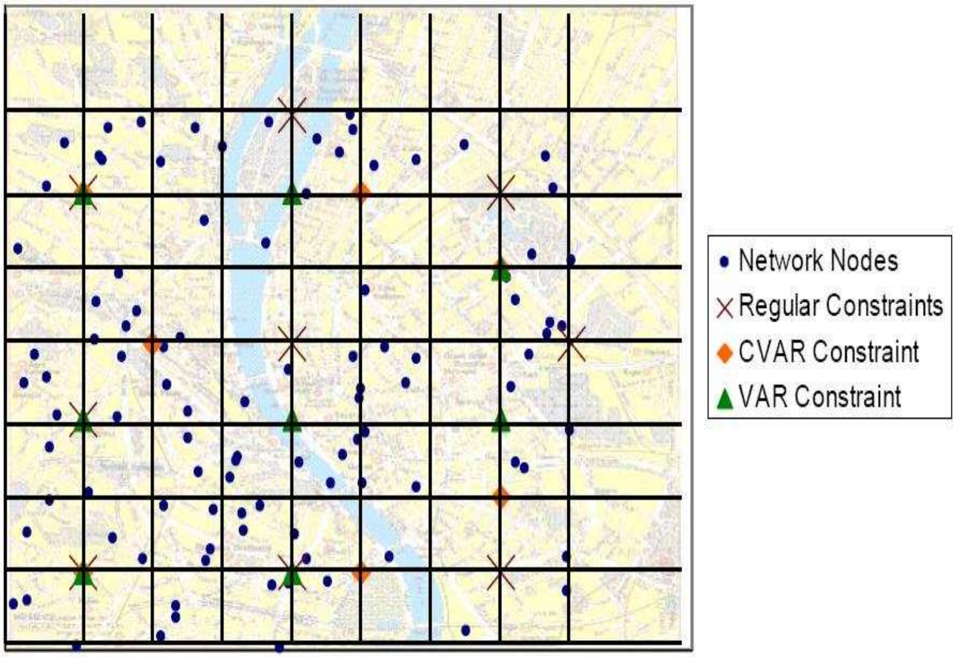


Figure 12: Case study 1 continued. The placement of jammers is shown when the problem is solved using VaR and CVaR constraints.

In this example, the VaR formulation requires 11% less jamming devices with almost the same quality as the formulation with the standard constraints. However, this formulation requires nearly 16 hours of computation time. The CVaR formulation gives a solution with a very good jamming quality and requires 22% less jamming devices than the standard formulation and 11% less devices than the VaR formulation. Furthermore, the CVaR formulation requires an order of magnitude less computing time than the formulation with VaR constraints.

6.1.2. *Connectivity Formulation.* We now present a case study where the WNJP was solved using the connectivity index formulation (CIP). The communication graph consists of 30 nodes and 60 edges. The maximal number of jamming devices available is 36. We set the maximal allowed connectivity index of any node to be 3. In Figure 13 we can see the original graph with the communication links prior to jamming. The result of the VaR and CVaR solutions is seen in Figure 14. The confidence level for both the VaR and CVaR formulations was 0.9. Both formulations provide optimal solutions for the given instance. The resulting computation time for the VaR formulation was 15 minutes 34 seconds, while the CVaR formulation required only 7 minutes 33 seconds.

6.2. **Heuristic for Jamming Under Complete Uncertainty.** Here, we describe the implementation of a randomized local search heuristic for the case of jamming under complete uncertainty. Recall that the subproblem for which the bounds in Theorem 2 and Theorem 3 we derived place n jamming devices, where n is a perfect square. The obvious drawback of this technique is the situation where for example $\underline{R}$ requires 25 jamming devices and $\bar{R}$ calls for 16, and the optimal solution to the general problem is 18. Using the uniform grid approach will require nearly 40% more devices than are needed to cover the region.

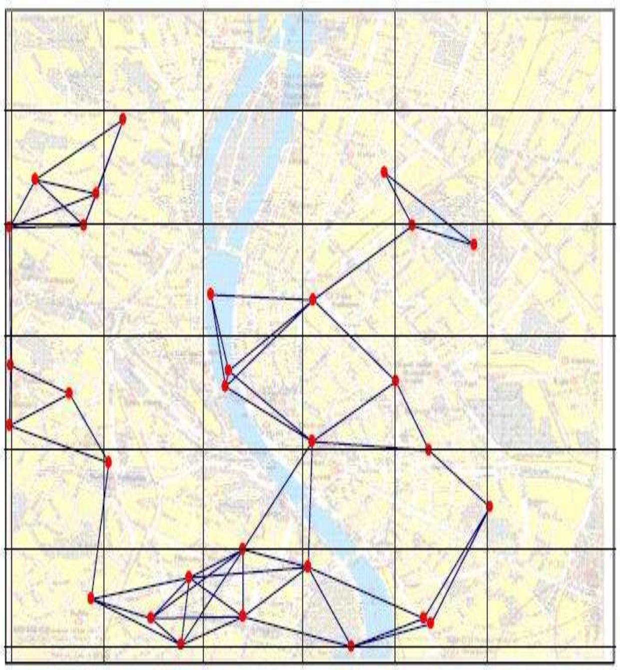
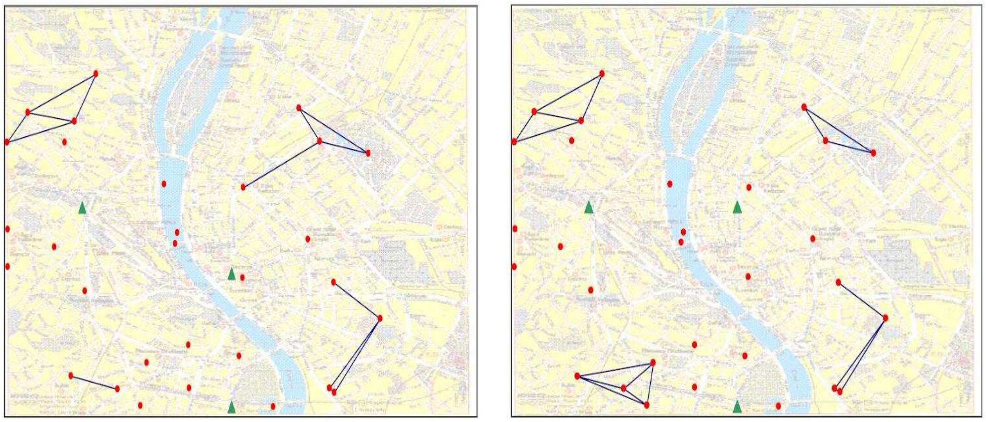


Figure 13: Case Study 2: Original graph.



(a) (b)

Figure 14: (a) VaR Solution. (b) CVaR Solution. In both cases, the triangles represent the jammer locations.

Pseudocode for the local search is given in Figure 15. The heuristic takes as input the size of the region containing the network (`region`). The number of jamming devices required to cover the area of `region` by the lower bound on the grid step (upper bound on jamming devices) derived in Theorem 2 (`ubJammers`) is the second input parameter. In line 1, the optimal solution (X^*) is set to `ubJammers`.

```
procedure randLocalSearch(region, ubJammers)
1   $X^* \leftarrow$ ubJammers
2   while stoppingCriteria = FALSE do
3     randScatter(region,$X^*$)
4     localOpt = FALSE
5     while localOpt = FALSE do
6       $\mathcal{P} \leftarrow$ leastJammedPoints(region)
7       moveJammers($\mathcal{P}$)
8     end
9     if allJammed = TRUE then
10      $X^* \leftarrow X^* - 1$
11    end
12  end
13  return $X^*$
end procedure randLocalSearch
```

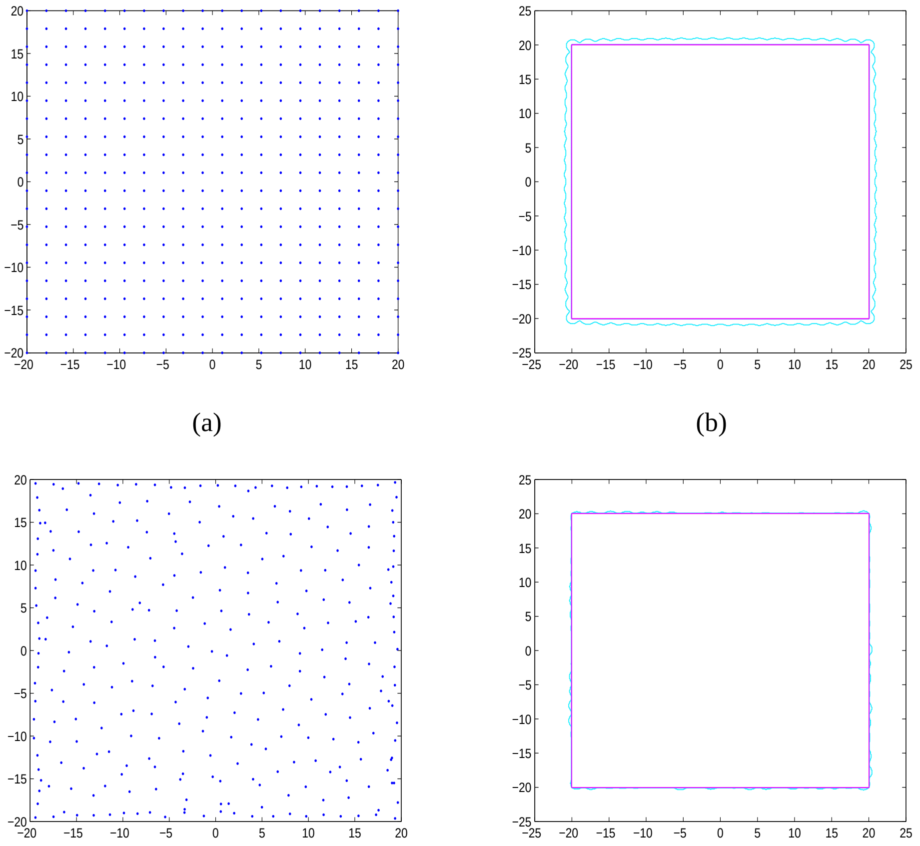
Figure 15: Pseudocode for the randomized local search for uncertain jamming.

The **while** loop from lines 2-12 is where the local optimization takes place. In line 3, the jamming devices are randomly scattered within the square region known to contain the network. Next in the **while** loop from lines 5-8, those points which are receiving the least amount of jamming energy are assigned to the set $\mathcal{P}$. Then, the jamming devices are moved along a gradient towards the points in $\mathcal{P}$ until these are points are covered. Several methods are available for the function `moveJammers` including the method of steepest descent [2] or the more efficient method of conjugated gradients [14, 15]. The heuristic then determines if all points have been jammed. If this is the case, then in line 9 we decrement the number of jamming devices by one and return to line 2. If all points are not jammed, we repeat the loop until either all points are covered or until a stopping criteria is met in which case we exit the **while** loop. The final value of X^* is returned as the solution in line 13.

The proposed method is still being tested and the full results will be reported in a paper which is currently in preparation and will appear later this year [4]. Preliminary results indicate that the heuristic solutions require 25% less jamming devices on average than the uniform grid approach, indicating a significant decrease in cost. An example comparison between the two approaches can be seen in Figure 16. For this example, a point requires 3 units of jamming energy before it is declared to be jammed. Figure 16(a) represents the placement of the jamming devices according to the uniform grid solution from Theorem 2. In this case, 400 devices are required. In Figure 16(b) we see the associated coverage from this solution. The scallop shell around the bounding box containing the network indicates that in fact, the entire area is jammed, but perhaps more devices are used than are necessary. In subfigure (c), we see the placement of the 298 jamming devices according to the heuristic solution. Notice in Figure 16(d) that the coverage outside the bounding box is reduced significantly while still jamming all points in the region. The heuristic reduces the required number of devices by 25.5%.

7. CONCLUDING REMARKS AND FUTURE RESEARCH

Using optimization approaches to jam wireless communication networks is a novel approach which was only recently introduced [5, 6]. There is still a great deal of work to be done on this problem which will help military strategists ensure the best level of performance against a hostile force. Below we briefly outline several extensions and areas in which future research can and is being performed.



(a) (b)

(c) (d)

Figure 16: (a) Device placement on uniform grid. The total number of jamming devices required is 400. (b) Coverage display of uniform placement. (c) Heuristic jammer placement. The total number of required devices is 298. (d) Heuristic coverage plot.

7.1. **Alternative Formulations.** A generalization of the node coverage formulation including uncertainties in the number of communication nodes and their coordinates might be considered. For the connectivity index problem, there might exist uncertainties in the number of network nodes, their locations, and the probability that a node will recover a jammed link.

Throughout our discussion, we have limited ourselves to cases in which the jamming devices and communication nodes have been equipped with omnidirectional antennas. We are currently considering a different formulation of the problem based on maximum network flows [1]. That is, consider the communication graph $G = (V, E)$ with a set U of arc capacities, a source node $s \in V$, and a sink node $t \in V$. Each node in the graph acts as both a server and client with the exception of s which only transmits and t which only receives data. Suppose further that each receiver is equipped with a k-sectored antenna. A sectored antenna is a set of directional antennas that can cover all directions but can isolate certain sectors. Thus, an arc can be jammed only by those jamming devices that are located in the same sector as the transmitter. Then the objective of the MAXIMUM NETWORK FLOW formulation is to find locations of jamming devices such that the expected maximum flow on the network is minimized.

We are considering this formulation with the incorporation of various uncertainties including the arc capacities, the number and coordinates of the communication nodes, type of sectored antennas, and the probability of nodes recovering after being jammed. Another formulation we are considering ha the objective of maximizing the lifetime of the jamming devices which are assumed to be equipped with a battery having a finite lifespan. The amount of battery power consumed is a function of the distance to the nodes receiving the majority of the jamming energy. The goal of the MAXIMUM LIFETIME formulation is to jam the network with the fewest jamming device while maximizing the life expectancy of the jamming devices. Obviously, there are many extensions which can be made and much work is yet to be done.

7.2. **Heuristics.** The inherent complexity of the aforementioned formulations motivates the need for efficient heuristics to solve real-world instances within reasonable computing times. Along with the local search previously mentioned, we are designing and implementing several other methods which will be reported in [4]. One method currently begin tested is a Greedy Randomized Adaptive Search Procedure (GRASP) [27] for the OPTIMAL NETWORK COVERING formulation. GRASP is an multi-start heuristic for combinatorial optimization which has been used with great success over the past decade on many problems including BROADCAST SCHEDULING [3], QUADRATIC ASSIGNMENT [21], and MAXIMUM CUT [9], to name a few. For an annotated bibliography of GRASP, the reader is referred to [10].

GRASP is a two-phase procedure which generates solutions through the controlled use of random sampling, greedy selection, and local search. For a given problem Π, let F be the set of feasible solutions for Π. Each solution $X \in F$ is composed of k discrete components $a_1, \ldots, a_k$. GRASP constructs a sequence $\{X\}_i$ of solutions for Π, such that each $X_i \in F$. The algorithm returns the best solution found after all iterations. The GRASP for ONC is still in the infancy stage; however, preliminary results are promising. In addition to the GRASP for ONC, we are planning to implement other heuristics for the ONC and CIP formulation such as Variable Neighborhood Search [24], Genetic Algorithm [13], Tabu Search [12], and Simulated Annealing [19]. These are only a few ideas and extensions that can be examined. We hope that our future endeavors will produce excellent results and will encourage others to investigate this new and interesting optimization problem.

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REFERENCES

[1] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice-Hall, 1993.

[2] D.P. Bertsekas. *Nonlinear Programming*. Athena Scientific, second edition, 1999.

[3] C.W. Commander, S.I. Butenko, and P.M. Pardalos. On the performance of heuristics for broadcast scheduling. In D. Grundel, R. Murphey, and P. Pardalos, editors, *Theory and Algorithms for Cooperative Systems*, pages 63–80. World Scientific, 2004.

[4] C.W. Commander, P.M. Pardalos, V. Ryabchenko, O. Shylo, and S. Uryasev. Recent advances in eavesdropping and jamming communication networks. In D.A. Grundel, R.A. Murphey, P.M. Pardalos, and O.A. Prokopyev, editors, *6th International Conference on Cooperative Control and Optimization*. World Scientific, in preparation and to appear 2006.

[5] C.W. Commander, P.M. Pardalos, V. Ryabchenko, S. Uryasev, and G. Zrazhevsky. Jamming communication networks under complete uncertainty. *Networks*, submitted, 2006.
[6] C.W. Commander, P.M. Pardalos, V. Ryabchenko, S. Uryasev, and G. Zrazhevsky. The wireless network jamming problem. *Journal of Combinatorial Optimization*, submitted, 2006.
[7] K.J. Cormican, D.P. Morton, and R.K. Wood. Stochastic network interdiction. *Operations Research*, 46(2):184–197, 1998.
[8] A. Farago. Graph theoretic analysis of ad hoc network vulnerability. In *Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt '03)*, 2003.
[9] P. Festa, P.M. Pardalos, M.G.C. Resende, and C.C. Ribeiro. Randomized heuristics for the MAX-CUT problem. *Optimization Methods and Software*, 7:1033–1058, 2002.
[10] P. Festa and M.G.C. Resende. Grasp: an annotated bibliography. In C.C. Ribeiro and P. Hansen, editors, *Essays and Surveys on Metaheuristics*, pages 325–367. Kluwer Academic Publishers, 2001.
[11] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company, 1979.
[12] F. Glover and M. Laguna. *Tabu Search*. Kluwer Academic Publishers, Boston, MA, 1997.
[13] D.E. Goldberg. *Genetic Algorithms in Search, Optimization, and Machine Learning*. Addison-Wesley, 1989.
[14] W.W. Hager and H. Zhang. A new conjugate graident method with guaranteed descent and an efficient line search. *SIAM Journal on Optimization*, 16:170–192, 2005.
[15] W.W. Hager and H. Zhang. Algorithm 851: CG_DESCENT, A conjugate gradient method with guaranteed descent. *ACM Transactions on Mathematical Software*, 32:113–137, 2006.
[16] G. Holton. *Value-at-Risk: Theory and Practice*. Academic Press, 2003.
[17] E. Israeli and R.K. Wood. Shortest-path network interdiction. *Networks*, 40(2):97–111, 2002.
[18] R. Kershner. The number of circles covering a set. *American Journal of Mathematics*, 61(3):665–671, 1939.
[19] S. Kirkpatrick, C. Gelatt, and M. Vecchi. Optimization by simulated annealing. *Science*, 220:671–680, 1983.
[20] D. Knuth. *The art of computer programming, volume 3: sorting and searching*, volume 3. Addison-Wesley, 1997.
[21] Y. Li, P.M. Pardalos, and M.G.C. Resende. A greedy randomized adaptive search procedure for the quadratic assignment problem. In P.M. Pardalos and H. Wolkowicz, editors, *Quadratic Assignment and Related Problems*, volume 16 of *DIMACS Series on DIscrete Mathematics and Theoretical Computer Science*, pages 237–261. 1994.
[22] C. Lim and J.C. Smith. Algorithms for discrete and continuous multicommodity flow network interdiction problems. *IIE Transactions*, To appear.
[23] C. Lim, J.C. Smith, and F. Sudargho. Survivable network design under optimal and heuristic interdiction scenarios. *Journal of Global Optimization*, submitted.
[24] N. Mladenović and P. Hansen. Variable neighborhood search. *Computers in Operations Research*, 24:1097–1100, 1997.
[25] G. Noubir. *On connectivity in ad hoc networks under jamming using directional antennas and mobility*, volume 2957 of *Lecture Notes in Computer Science*, pages 186–200. Springer, 2004.
[26] M.G.C. Resende and P.M. Pardalos. *Handbook of Optimization in Telecommunications*. Springer, 2006.
[27] M.G.C. Resende and C.C. Ribeiro. Greedy randomized adaptive search procedures. In F. Glover and G. Kochenberger, editors, *Handbook of Metaheuristics*, pages 219–249. Kluwer Academic Publishers, 2003.
[28] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *The Journal of Risk*, 2(3):21–41, 2000.
[29] S. Uryasev. Conditional value-at-risk: Optimization algorithms and applications. *Financial Engineering News*, 14:1–5, 2000.
[30] K. Wood. Deterministic network interdiction. *Mathematical and Computer Modeling*, 17(2):1–18, 1993.

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