# Efficient Execution in the Secondary Mortgage Market: A Stochastic Optimization Model Using *CVaR* Constraints

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**Abstract**

Efficient execution is a significant task faced by mortgage bankers attempting to profit from the secondary market. The challenge of efficient execution is to sell or securitize a large number of heterogeneous mortgages in the secondary market in order to maximize expected revenue under a risk tolerance. This paper develops a stochastic optimization model to perform efficient execution that considers secondary marketing functionality including loan-level efficient execution, guarantee fee buy-up or buy-down, servicing retain or release, and excess servicing fee. Since efficient execution involves random cash flows, lenders must balance between expected revenue and risk. We employ a $CVaR$ risk measure in this efficient execution model that maximizes expected revenue under a $CVaR$ constraint. By solving the efficient execution problem under different risk tolerances specified by a $CVaR$ constraint, an efficient frontier could be found. The model is formulated as a mixed 0-1 linear programming problem. A case study shows that realistic instances of the efficient execution problem can be solved in acceptable time (approximately one minute) with CPLEX-90 solver on a PC.

**Keywords:** *efficient execution, secondary mortgage market, mortgage-backed security (MBS), Fannie Mae, conditional value-at-risk (CVaR).*

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## 1. Introduction

Mortgage banks (or lenders) originate mortgages in the primary market. Besides keeping the mortgages as a part of the portfolio, a lender may sell the mortgages to mortgage buyers (or conduits) or securitize the mortgages as mortgage-backed securities (MBSs) through MBS swap programs in the secondary market. In the United States, three government-sponsored enterprises (GSEs) (Fannie Mae, Freddie Mac, and Ginnie Mae) provide MBS swap programs in which mortgage bankers can deliver their mortgages into appropriate MBS pools in exchange for MBSs.

In practice, most mortgage bankers prefer to participate in the secondary market based on the following reasons. First, mortgage banks would get funds from secondary marketing and then use the funds to originate more mortgages in the primary market and earn more origination fees. Second, the value of a mortgage is risky and depends on several sources of uncertainties, i.e., default risk, interest rate risk, and prepayment risk. Mortgage bankers could reduce risks by selling or securitizing mortgages in the secondary market. More exactly, when mortgages are sold as a whole loan, all risks would be transferred to mortgage buyers. On the other hand, when mortgages are securitized as MBSs, the risky cash flows of mortgages are split into guarantee fees, servicing fees, and MBS coupon payments, which belong to MBS issuers, mortgage servicers, and MBS investors, respectively. In this case, mortgage bankers are exposed only to risk from retaining the servicing fee and other risky cash flows are transferred to different parties.

A significant task faced by mortgage bankers attempting to profit from the secondary market is efficient execution. The challenge of efficient execution is to sell or securitize a large number of heterogeneous mortgages in the secondary market in order to maximize expected revenue through complex secondary marketing functionality. In addition, to deal with the uncertain cash flows from the retained servicing fee, the balance between mean revenue and risk is also an important concern for mortgage bankers.

This paper develops a stochastic optimization model to perform an efficient execution that considers secondary marketing functionality, including loan-level efficient execution, guarantee fee buy-up or buy-down, servicing retain or release, and excess servicing fee. Further, we employ Conditional Value-at-Risk ($CVaR$), proposed by Rockafellar and Uryasev (2000), as a risk measure in the efficient execution model that maximizes expected revenue under a $CVaR$ constraint. By solving the efficient execution problem under different risk tolerances specified by a $CVaR$ constraint, an efficient frontier could be found.

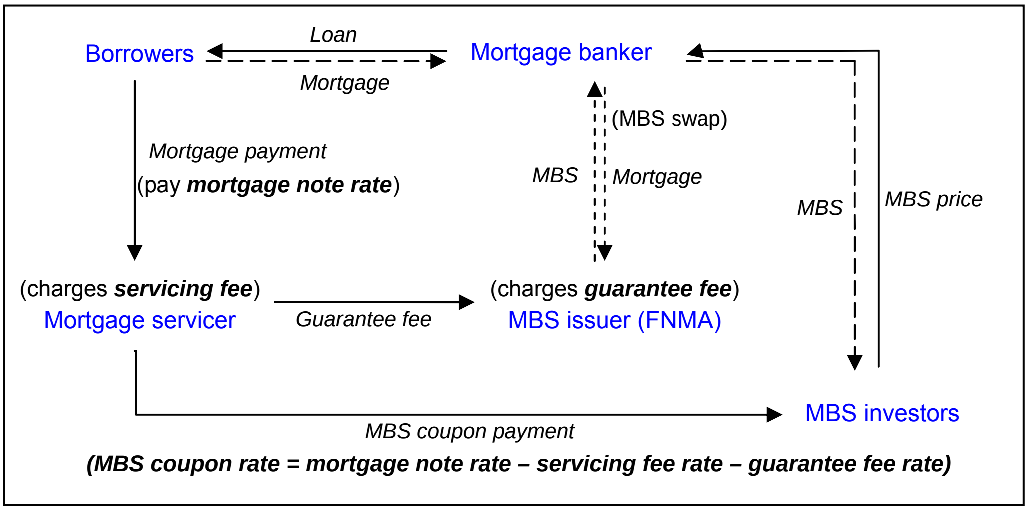
A great deal of research has focused on mortgage valuation (Kau, Keenan, Muller, and Epperson (1992); Kau (1995); Hilliard, Kau and Slawson (1998); and Downing, Stanton, and Wallace (2005)), MBS valuation (Schwartz and Torous (1989); Stanton (1995); Sugimura (2004)), and mortgage servicing right valuation (Aldrich, Greenberg, and Payner (2001); Lin, Chu, and Prather (2006)). However, academic literature addressing topics of mortgage secondary marketing is scant. Hakim, Rashidian, and Rosenblatt (1999) addressed the issue of fallout risk, which is an upstream secondary marketing problem. To the best of our knowledge, we have not seen any literature focusing on efficient execution.

The organization of this paper is as follows: Section 2 discusses mortgage securitization. We describe the relationship between MBS market participants and introduce the Fannie Mae MBS swap program. Section 3 presents our model development. Section 4 reports our results, and the final section presents our conclusions.

## 2. Mortgage Securitization

Mortgage bankers may sell mortgages to conduits at a price higher than the par value[3] to earn revenue from the whole loan sales. However, for lenders who possess efficient execution knowledge, mortgage securitization through MBS swap programs of GSEs may bring them higher revenue than the whole loan sale strategy. This paper considers pass-through MBS swap programs provided by Fannie Mae (FNMA). To impose considerations of MBS swap programs of other GSEs is straightforward.

**Figure 2.1: The relationship between participants in the pass-through MBS market.** Mortgage bankers originate mortgage loans by signing mortgage contracts with borrowers who commit to making monthly payments with a fixed interest rate known as the mortgage note rate. To securitize those mortgages, mortgage bankers deliver the mortgages into an MBS swap in exchange for MBSs. Further, mortgage bankers sell the MBSs to MBS investors and receive MBS prices in return. The MBS issuer provides MBS insurance and charges a base guarantee fee. Mortgage servicers provide mortgage servicing and a base servicing fee is disbursed for the servicing. Both fees are a fixed percentage (servicing fee rate or guarantee fee rate) of the outstanding mortgage balance and decline over time as the mortgage balance amortizes. Deducting guarantee fees and servicing fees from mortgage payments, the remaining cash flows that pass-through to the MBS investors are known as MBS coupon payments with a rate of return equal to the mortgage note rate minus the servicing fee rate minus the guarantee fee rate.




[3] Mortgage bankers underwrite mortgages at a certain mortgage note rate. The par value is the value of the mortgage when the discount interest rate equals the mortgage note rate. In other words, the par value of a mortgage is its initial loan balance.

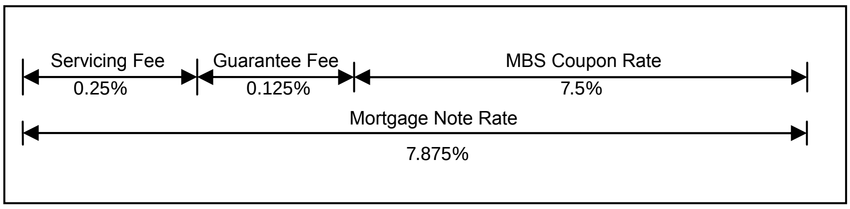
This section describes the relationship between participants in the pass-through MBS market and detail the procedure of mortgage securitization through a MBS swap program.

Participants in the MBS market can be categorized into five groups: borrowers, mortgage bankers, mortgage servicers, MBS issuers, and mortgage investors. The relationship between these five participants in the pass-through MBS market is shown in Figure 2.1.

In Figure 2.1, solid lines show cash flows between participants and dashed lines represent mortgage contracts and MBS instruments between them. Mortgage bankers originate mortgage loans by signing mortgage contracts with borrowers who commit to making monthly payments in a fixed interest rate known as the mortgage note rate. To securitize those mortgages, mortgage bankers deliver the mortgages into an MBS swap in exchange for MBSs. Further, mortgage bankers sell the MBSs to MBS investors and receive MBS prices in return.

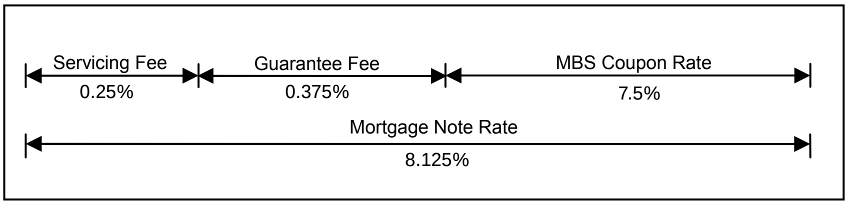
MBS issuers provide MBS insurance to protect the MBS investors against default losses and charge a base guarantee fee, which is a fixed percentage, known as the guarantee fee rate, of the outstanding mortgage balance, and which declines over time as the mortgage balance amortizes. Mortgage bankers negotiate the base guarantee fee rate with Fannie Mae and have the opportunity to "buy-down" or "buy-up" the guarantee fee. When lenders buy-down the guarantee fee, the customized guarantee fee rate is equal to the base guarantee fee rate minus the guarantee fee buy-down spread. Further, lenders have to make an upfront payment to Fannie Mae. On the other hand, the buy-up guarantee fee allows lenders to increase the guarantee fee rate from the base guarantee fee rate and receive an upfront payment from Fannie Mae. For example, if a lender wants to include a 7.875% mortgage with a 0.25% base guarantee fee and a 0.25% base servicing fee in a 7.5% pass-through MBS (Figure 2.2), the lender can buy-down the guarantee fee rate to 0.125% from 0.25% by paying Fannie Mae an upfront amount equal to the present value of the cash flows of the 0.125% difference and maintaining the 0.25% base servicing fee.

**Figure 2.2: Guarantee fee buy-down.** A lender may include a 7.875% mortgage in a 7.5% pass-through MBS by buying-down the guarantee fee to 0.125% from 0.25% and maintaining the 0.25% base servicing fee.




If a lender chooses to include an 8.125% mortgage in the 7.5% pass-through MBS (Figure 2.3), the lender can buy-up the guarantee fee by 0.125% in return for a present value of the cash flows of the 0.125% difference. The buy-down and buy-up guarantee fee features allow lenders to maximize the present worth of revenue.

**Figure 2.3: Guarantee fee buy-up.** A lender may include an 8.125% mortgage in a 7.5% pass-through MBS by buying-up the guarantee fee to 0.375% from 0.25% and maintaining the 0.25% base servicing fee.




Mortgage servicers provide mortgage servicing, including collecting monthly payments from borrowers, sending payments and overdue notices, maintaining the principal balance report, etc. A base servicing fee is disbursed for the servicing, which is a fixed percentage, known as the base servicing fee rate, of the outstanding mortgage balance, and which declines over time as the mortgage balance amortizes. Mortgage bankers have the servicing option to sell the mortgage servicing (bundled with the base servicing fee) to a mortgage servicer and receive an upfront payment from the servicer or retain the base servicing fee and provide the mortgage servicing.

Deducting guarantee fees and servicing fees from mortgage payments, the remaining cash flows that pass-through to the MBS investors are known as MBS coupon payments, which contain a rate of return known as the MBS coupon rate (or pass-through rate), equal to the mortgage note rate minus the servicing fee rate minus the guarantee fee rate.

Fannie Mae purchases and swaps more than 50 types of mortgages on the basis of standard terms. This paper focuses on pass-through MBS swaps of 10-, 15-, 20-, and 30-year fixed-rate mortgages. Mortgages must be pooled separately by the time to maturity. For instance, 30-year fixed-rate mortgages are separated from 15-year fixed-rate ones. For each maturity, Fannie Mae provides different MBS pools characterized by MBS coupon rates that generally trade on the half percent (4.5%, 5.0%, 5.5%, etc.). Mortgage lenders have the option to deliver individual mortgages into one of these eligible MBS pools, which allows lenders to maximize revenue.

Further, the mortgage note rate must support the MBS coupon rate plus the servicing fee rate plus the guarantee fee rate. Therefore, when securitizing a mortgage as an MBS, mortgage bankers have to manipulate the servicing fee rate and guarantee fee rate so that Equation (1) is satisfied.

$$\textit{Mortgage Note Rate} = \textit{Servicing Fee Rate} + \textit{Guarantee Fee Rate} + \textit{MBS Coupon Rate}. \tag{1}$$

Mortgage bankers could retain an excess servicing fee from the mortgage payment, which, like the base servicing fee, is a fixed percentage, known as the excess servicing fee rate, of the outstanding mortgage balance and which declines over time as the mortgage balance amortizes. In Equation (2), the excess servicing fee rate is equal to the excess of the mortgage note rate over the sum of the MBS coupon rate, customized guarantee fee rate, and base servicing fee rate. In other words, the servicing fee rate in Equation (1) consists of the base servicing fee rate and the excess servicing fee rate.

$$\text{Excess Servicing Fee Rate} \\ = \text{Mortgage Note Rate - MBS Coupon Rate - Guarantee Fee Rate – Base Servicing Fee Rate.} \tag{2}$$

In the example shown in Figure 2.3, the excess servicing fee may be sold to Fannie Mae by buying-up the guarantee fee. Another option for mortgage bankers is to retain the excess servicing fee in their portfolio and to receive cash flows of the excess servicing fee during the life of the mortgage. The value of the excess servicing fee is equal to the present value of its cash flows. This value is stochastic since borrowers have the option to terminate mortgages before maturity and the interest rate used to discount the future cash flows is volatile. Therefore, efficient execution becomes a stochastic optimization problem. Similar to the guarantee fee buy-up and buy-down features, the excess servicing fee allows lenders to maximize the expected revenue.

# 3. Model

Efficient execution is a central problem of mortgage secondary marketing. Mortgage bankers originate mortgages in the primary market and execute the mortgages in the secondary market to maximize their revenue through different secondary marketing strategies. In the secondary market, each mortgage can be executed in two ways, either sold as a whole loan, or pooled into an MBS with a specific coupon rate. When mortgages are allocated into an MBS pool, we consider further the guarantee fee buy-up/buy-down option, the mortgage servicing retain/release option, and excess servicing fee to maximize the total revenue.

Based on secondary marketing strategies, mortgage bankers may retain the base servicing fee and the excess servicing fee when mortgages are securitized. The value of the retained servicing fee is random and affected by the uncertainty of interest rate term structure and prepayment. Therefore, a stochastic optimization model is developed to maximize expected revenue under a risk tolerance and an efficient frontier can be found by optimizing expected revenue under different risk tolerances specified by a risk measure.

## 3.1 Risk Measure

Since Markowitz (1952), variance (and covariance) has become the predominant risk measure in finance. However, the risk measure is suited only to the case of elliptic distributions, like normal or t-distributions with finite variances (Szegö (2002)). The other drawback of variance risk measure is that it measures both upside and downside risks. In practice, finance risk managers are concerned only with the downside risk in most cases.

A popular downside risk measure in economics and finance is Value-at-Risk (*VaR*) (Jorion (2000)), which measures α percentile of loss distribution. However, as was shown by Artzner *et al.* (1999), *VaR* is ill-behaved and non-convex for general distribution. The other disadvantage of *VaR* is that it only considers risk at α percentile of loss distribution and does not consider how much worse the $\alpha$-tail (the worst 1-$\alpha$ percentage of scenarios) could be.

To address this issue, Rockafellar and Uryasev (2000, 2002) proposed Conditional Value-at-Risk (*CVaR*), which is the mean value of $\alpha$-tail of loss distribution. It has been shown that *CVaR* satisfies the axioms of coherent risk measures proposed by Artzner *et al.* (1999) and has desirable properties. Most importantly, Rockafellar and Uryasev (2000) showed that *CVaR* constraints in optimization problems can be expressed by a set of linear constraints and incorporated into problems of optimization.

The paper uses *CVaR* as the measure of risk in developing the efficient execution model maximizing expected revenue under a *CVaR* constraint. Thanks to *CVaR*, an efficient execution model could be formulated as a mixed 0-1 linear programming problem.

It is worth mentioning that the prices of MBSs, prices of whole loan sale mortgages, upfront payment of released servicing, and upfront payment of guarantee fee buy-up or buy-down are deterministic numbers that can be observed from the secondary market. However, the revenue from retained base servicing fees and excess servicing fees are equal to the present value of cash flows of the fees. Because of the randomness of interest rate term structure and prepayment, the revenue from the fees is varied with different scenarios. Lenders could simulate scenarios based on their own interest rate model and prepayment model. The paper treats the scenarios as input data.

## 3.2 Model Development

This subsection presents the stochastic optimization model. The objective of the model is to maximize the expected revenue from secondary marketing. Four sources of revenue are included in the model: revenue from MBSs or whole loan sale, expected revenue from the base servicing fee, expected revenue from the excess servicing fee, and revenue from the guarantee fee buy-up/buy-down.

*(1) Revenue from MBSs or whole loan sale:*

$$f_1(z_c^m, z_w^m) = \sum_{m=1}^{M}\left[ L^m \times \sum_{c=1}^{C^m}\left(P_c^{t^m} \times z_c^m\right) + P_w^m \times z_w^m \right], \tag{3}$$

where

$M$ = total number of mortgages,

$m$ = index of mortgages ($m$ = 1, 2, …, $M$),

$L^m$ = loan amount of mortgage $m$,

$C^m$ = number of possible MBS coupon rates of mortgage $m$,

$c$ = index of MBS coupon rate,

$P_c^{t^m}$ = price of MBS with coupon rate index $c$ and maturity of $t^m$,

$t^m$ = maturity of mortgage $m$,

$P_w^m$ = whole loan sale price of mortgage $m$,

$$z_c^m = \begin{cases} 1, \text{if mortgage m is pooled into MBS with coupon rate index } c, \\ 0, \text{otherwise}, \end{cases}$$

$$z_w^m = \begin{cases} 1, \text{if mortgage } m \text{ is sold as a whole loan}, \\ 0, \text{otherwise}. \end{cases}$$

Equation (4) enforces that each mortgage could be either sold as a whole loan or delivered into a specific MBS pool:

$$\sum_{c=1}^{C^m} z_c^m + z_w^m = 1 \cdot \tag{4}$$

If mortgage *m* is securitized as an MBS with coupon rate index $\hat{c}$, then $z_{\hat{c}}^m = 1$, $z_c^m = 0$ for all $c \neq \hat{c}$, and $z_w^m = 0$, and the revenue from mortgage *m* equals $L^m \times P_{\hat{c}}^{\prime m} \times z_{\hat{c}}^m$. On the other hand, if mortgage *m* is sold as a whole loan, then $z_c^m = 0$ for all $c$, and $z_w^m = 1$, and the revenue from mortgage *m* equals $P_w^m \times z_w^m$. The total revenue from the *M* mortgages is shown in Equation (3).

*(2) Expected revenue from the base servicing fee of securitized mortgages:*

$$f_2(z_{sbo}^m, z_{sbr}^m) = \sum_{m=1}^{M} \left(L^m \times B^m \times z_{sbo}^m\right) + \sum_{m=1}^{M} \left( \sum_{k=1}^{K} p_k \left(L^m \times R_{sb}^m \times K_{sr}^{mk}\right) - c_s^m \right) z_{sbr}^m , \tag{5}$$

where

$$z_{sbr}^m = \begin{cases} 1, \text{if the servicing of mortgage } m \text{ is retained}, \\ 0, \text{otherwise}, \end{cases}$$

$$z_{sbo}^m = \begin{cases} 1, \text{if the servicing of mortgage } m \text{ is sold}, \\ 0, \text{otherwise}, \end{cases}$$

$B^m$ = base servicing value of mortgage *m*,
$p_k$ = the probability of scenario *k*,
$K$ = number of scenarios,
$c_s^m$ = servicing cost of mortgage *m*,
$K_{sr}^{mk}$ = retained servicing fee multiplier of mortgage *m* under scenario *k*.

The retained servicing multipliers $K_{sr}^{mk}$ for scenario $k$ could be generated by simulation. Mortgage bankers can simulate the random discounted cash flows of the retained servicing fee by using their own interest rate model and prepayment model. Then, a retained servicing multiplier $K_{sr}^{mk}$ could be found associated with each scenario. In this paper, we treat the retained servicing multiplier $K_{sr}^{mk}$ as input data. Details of how to get the retained servicing multiplier $K_{sr}^{mk}$ is beyond the scope of this paper.

Equation (6) enforces that when a mortgage is securitized, the servicing of the mortgage can be either released or retained, and the revenue from mortgage servicing exists only if the mortgage is securitized as an MBS instead of being sold as a whole loan:

$$z_w^m + z_{sbo}^m + z_{sbr}^m = 1. \tag{6}$$

More exactly, if mortgage $m$ is sold as a whole loan, then $z_w^m = 1$, $z_{sbo}^m = z_{sbr}^m = 0$, and the revenue from the base servicing fee equals zero. On the other hand, if mortgage $m$ is securitized as an MBS and the servicing of mortgage $m$ is sold, then $z_{sbo}^m = 1$, $z_{sbr}^m = 0$, $z_w^m = 0$, and the upfront payment from the mortgage servicer equals $L^m \times B^m \times z_{sbo}^m$; otherwise, $z_{sbr}^m = 1$, $z_{sbo}^m = 0$, $z_w^m = 0$, and the expected revenue from the released servicing equals the expected revenue from base servicing fee $\sum_{k=1}^{K} p^k \left(L^m \times R_{sb}^m \times K_{sr}^{mk} \times z_{sbr}^m\right)$ minus servicing cost $c_s^m \times z_{sbr}^m$. The total expected revenue from the $M$ mortgages is expressed in Equation (5).

*(3) Expected revenue from the excess servicing fee of securitized mortgages:*

$$f_3(r_{ser}^m) = \sum_{m=1}^{M} \sum_{k=1}^{K} p_k \left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right), \tag{7}$$

where $r_{ser}^m$ = retained excess servicing fee of mortgage *m.*

If mortgage $m$ is securitized as an MBS, the mortgage generates expected revenue $\sum_{k=1}^{K} p_k \left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right)$ from retaining the excess servicing fee. The total expected revenue from the $M$ mortgages is shown in Equation (7).

*(4) Revenue from the guarantee fee buy-up/buy-down of securitized mortgages:*

$$f_4(r_{gu}^m, r_{gd}^m) = \sum_{m=1}^{M} \left(K_u^m \times L^m \times r_{gu}^m\right) - \sum_{m=1}^{M} \left(K_d^m \times L^m \times r_{gd}^m\right), \tag{8}$$

where

$r_{gu}^m$ = guarantee fee buy-up spread of mortgage *m*,
$r_{gd}^m$ = guarantee fee buy-down spread of mortgage *m*,
$K_u^m$ = guarantee fee buy-up multiplier of mortgage *m*,
$K_d^m$ = guarantee fee buy-down multiplier of mortgage *m.*

Guarantee fee buy-up and buy-down multipliers (or ratios) $K_u^m$ and $K_d^m$, announced by Fannie

Mae, are used to calculate the upfront payment of a guarantee fee buy-up and buy-down. Lenders buy-up the guarantee fee of mortgage *m* to receive an upfront $K_u^m \times L^m \times r_{gu}^m$ from Fannie Mae. On the other hand, they can buy-down the guarantee fee of mortgage *m* and make an upfront payment $K_d^m \times L^m \times r_{gd}^m$ to Fannie Mae. The total revenue from the *M* mortgages is shown in Equation (8).

Guarantee fee buy-up/buy-down and retaining the excess servicing fee are considered only when mortgage *m* is securitized. Equation (9) enforces $r_{ser}^m$, $r_{gu}^m$, and $r_{gd}^m$ to be zero when mortgage *m* is sold as a whole loan,

$$z_w^m + r_{gu}^m + r_{gd}^m + r_{ser}^m \leq 1, \tag{9}$$

From equation (1), when mortgages are securitized as MBSs, the mortgage note rate has to support the MBS coupon rate, servicing fee rate, and guarantee fee rate. Equation (10) places a mathematic expression in the restriction.

$$\sum_{c=1}^{C^m} R_c z_c^m + r_{gu}^m - r_{gd}^m + r_{ser}^m \leq R_n^m - R_{sb}^m - R_{gb}^m, \tag{10}$$

where

$R_c$ = MBS coupon rate related to index *c*,

$R_n^m$ = note rate of mortgage *m*,

$R_{sb}^m$ = base servicing fee of mortgage *m*,

$R_{gb}^m$ = base guarantee fee of mortgage *m*,

Next, we introduce the *CVaR* constraint

$$CVaR_\alpha(L) \leq U, \tag{11}$$

where *L* is the loss function, $\alpha$ is the percentile of $CVaR$, and *U* is the upper bound of $CVaR$ losses. Equation (11) restricts the average of $\alpha$-tail of loss distribution to be less than or equal to *U*. In other words, the average losses of the worst 1-$\alpha$ percentage of scenarios should not exceed *U*. It is worth mentioning that *CVaR* is defined on a loss distribution. Therefore, we should treat revenue as negative losses when we use *CVaR* constraint in maximum revenue problem.

Rockafellar & Uryasev (2000) proposed that *CVaR* constraints in optimization problems can be expressed by a set of linear constraints.

$$\zeta + (1-\alpha)^{-1} \sum_{k=1}^{K} p_k z_k \leq U, \tag{12}$$

$$z_k \geq L_k - \zeta \quad \forall k = 1,...,K\ , \tag{13}$$

$$z_k \geq 0 \qquad \forall k = 1,...,K\ , \tag{14}$$

$$\begin{aligned} L_k = &-\sum_{m=1}^{M}\left[L^m \times \sum_{c=1}^{C^m}\left(P_c^m \times z_c^m\right) + P_{whole}^m \times z_w^m\right] - \sum_{m=1}^{M}\left(L^m \times B^m \times z_{sbo}^m\right) - \sum_{m=1}^{M}\left(L^m \times R_{sb}^m \times K_{sr}^{mk} \times z_{sbr}^m\right) \\ &-\sum_{m=1}^{M}\left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right) - \sum_{m=1}^{M}\left(K_u^m \times L^m \times r_{gu}^m\right) + \sum_{m=1}^{M}\left(K_d^m \times L^m \times r_{gd}^m\right). \end{aligned} \tag{15}$$

where $L_k$ is loss value in scenario $k$, and $\zeta$ and $z_k$ are real variables. Users of the model could specify their risk preference by selecting the value of $\alpha$ and $U$.

More constraints could be considered in the model based on the mortgage banker's preference. For instance, mortgage bankers would like to control the retained excess servicing fee based on the future capital demand or risk consideration. The excess servicing fee could be limited by an upper bound in three levels, including aggregate level, group level, and loan level. Constraint (16) limits average excess servicing fee across all mortgages by an upper bound $U_{se}^a$:

$$\sum_{m=1}^{M} L^m \times r_{se}^m \leq U_{se}^a \left(\sum_{m=1}^{M} L^m (1 - z_w^m)\right). \tag{16}$$

We further categorized mortgages into groups according to the year to maturity. Constraint (17) limits the group-level average excess servicing fee by an upper bound $U_{se}^j$ for each group $j$.

$$\sum_{m \in j} L^m \times r_{se}^m \leq U_{se}^j \left(\sum_{m \in j} L^m (1 - z_w^m)\right). \tag{17}$$

In addition, constraint (18) restrict the loan level excess servicing fee to an upper bound $U_{se}^m$:

$$0 \leq r_{ser}^m \leq U_{se}^m\ . \tag{18}$$

Furthermore, constraints (19) and (20) impose an upper bound $U_{gu}^m$ and $U_{gd}^m$ in the guarantee fee buy-up and buy-down spreads, respectively:

$$0 \leq r_{gu}^m \leq U_{gu}^m\ , \tag{19}$$

$$0 \leq r_{gd}^m \leq U_{gd}^m\ . \tag{20}$$

Those upper bounds are determined by the restrictions of an MBS swap program or sometimes by the decision of mortgage bankers. For example, the maximum guarantee fee buy-down spread accepted by Fannie Mae is the base guarantee fee rate.

Finally, we impose non-negativity constraints

$$r_{ser}^{m}, r_{gu}^{m}, r_{gd}^{m} \geq 0, \tag{21}$$

and binary constraints

$$z_{c}^{m}, z_{whole}^{m}, z_{sbr}^{m}, z_{sbo}^{m} \in \{0,1\} \qquad \forall m = 1, 2, ...., C^{m} . \tag{22}$$

The notations and model formulation are summarized in the Appendix.

# 4. Case Study

This section presents a case study. First, we discuss the data set of mortgages, MBSs, base servicing fee and guarantee fee multipliers, and scenarios of retained servicing fee multiplier. Next, we introduce the solver that is used to solve the mixed 0-1 linear programming problem. Then, we show the results including efficient frontier and sensitivity analysis.

## 4.1 Input Data

In the case study, we consider executing 1,000 fixed-rate mortgages in the secondary market. For each mortgage, the data includes years to maturity (YTM), loan amount, note rate, and guarantee fee. Table 4.1 summarizes the data on mortgages. Mortgages are categorized into four groups according to YTM. For each group, the table shows number of mortgages, and the minimum, mean, and maximum value of loan amount, note rate, and guarantee fee.

**Table 4.1 Summary of data on mortgages**

| | # of | Loan Amount ($) | | | | Note Rate (%) | | | Guarantee Fee (%) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| YTM | Mortgages | Min | Mean | Max | Sum | Min | Mean | Max | Min | Mean | Max |
| 10 | 13 | $539,323 | $164,693 | $330,090 | $2,141,009 | 4.375 | 5 | 5.75 | 0.125 | 0.1463 | 0.4018 |
| 15 | 148 | $34,130 | $172,501 | $459,000 | $25,530,131 | 4 | 5.2948 | 8 | 0.125 | 0.1867 | 0.8 |
| 20 | 11 | $38,920 | $137,024 | $291,320 | $1,507,260 | 5.125 | 5.9659 | 7.25 | 0.125 | 0.2115 | 0.8 |
| 30 | 828 | $24,000 | $194,131 | $499,300 | $160,740,083 | 4.75 | 5.9235 | 7.875 | 0.125 | 0.2124 | 1.05 |

MBS pools are characterized by the MBS coupon rate and YTM. In this case study, we consider 13 possible MBS coupon rates from 3.5% to 9.5%, increasing in increments of 0.5%, and four different YTM: 10, 15, 20, and 30 years. There are a total of 50 MBS pools since the MBS pool with a 3.5% coupon rate is not available for 20-year and 30-year mortgages. Table 4.2 shows the prices of MBSs for different MBS pools.

**Table 4.2 Summary of data on MBS prices of MBS pools**

| | MBS coupon rate (%) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| YTM | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 |
| 10 | 107.69 | 107.69 | 107.69 | 107.66 | 107.31 | 106.94 | 105.84 | 104.94 | 103.69 | 102.45 | 101.13 | 99.703 | 93.381 |
| 15 | 107.69 | 107.69 | 107.69 | 107.66 | 107.31 | 106.97 | 105.84 | 104.63 | 103.19 | 101.5 | 99.688 | 97.531 | 93.381 |
| 20 | 107.73 | 107.73 | 107.72 | 107.63 | 106.69 | 105.78 | 104.69 | 103.64 | 101.78 | 99.719 | 97.047 | 90.263 | N/A |
| 30 | 107.73 | 107.73 | 107.72 | 107.63 | 106.69 | 105.78 | 104.5 | 103.13 | 100.91 | 98.469 | 95.313 | 90.263 | N/A |

In the case study, we assume the base servicing rate is 25 bp (1bp = 0.01%) for all mortgages. The base servicing value is 1.09 for mortgages with maturity of 10, 15, and 20 years and 1.29 for mortgages with maturity of 30 years. In addition, we assume the servicing cost equal to zero. The guarantee fee buy-up and buy-down multipliers are summarized in Table 4.3, which shows that guarantee fee buy-up and buy-down multipliers depend on mortgage note rate and maturity. A mortgage with a higher note rate and longer maturity has larger multipliers. In addition, buy-down multipliers are larger than the buy-up multipliers under the same note rate and maturity.

In the case study, we consider 20 scenarios that are uniformly distributed across the range between 0 and $2K_{sr}^{m}$, where $K_{sr}^{m}$ is the expected retained servicing fee multiplier for mortgage *m* summarized in Table 4.3.

**Table 4.3 Guarantee fee buy-up and buy-down and expected retained servicing multipliers**

| Note | 30-YR | | 20-YR | | 15-YR | | 10-YR | | 30-YR | 10-, 15-, 20- YR |
|---|---|---|---|---|---|---|---|---|---|---|
| Rate | Buy-up | Buy-down | Buy-up | Buy-down | Buy-up | Buy-down | Buy-up | Buy-down | Expected Retained Servicing | Expected Retained Servicing |
| 4 | 5.65 | 7.6 | 4.795 | 6.522 | 3.6 | 5.55 | 3.418 | 5.268 | 5.75 | 4 |
| 5 | 4.95 | 6.9 | 4.2 | 5.927 | 2.95 | 4.9 | 2.8 | 4.65 | 5.74 | 3.6 |
| 6 | 3.15 | 5 | 2.67 | 4.407 | 1.5 | 3.287 | 1.376 | 3.176 | 4.19 | 2.76 |
| 7 | 1.65 | 3.3 | 1.395 | 2.995 | 0.95 | 2.75 | 0.9 | 2.7 | 2.22 | 2 |
| 8 | 0.95 | 2.5 | 0.75 | 2.35 | 0.55 | 2.35 | 0.496 | 2.296 | 1.6 | 0.62 |

Equation (23) defines the probability mass function for the variable of retained servicing fee multiplier.

$$f_X(x) = \begin{cases} 0.05, & x \in \{S : S = (.05 + .1z)K_{sr}^{m}, 0 \le z \le 19, z \in Integer\}, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

From Table 4.3, the expected retained servicing fee multiplier for mortgage *m* depends on note rate and maturity. A mortgage with a larger note rate and longer maturity has a higher expected retained servicing fee multiplier.

## 4.2 Solver

We used CPLEX-90 to solve the large-scale mixed 0-1 linear programming problem on an Intel Pentium 4, 2.8GHz PC. The running times for solving the instances of the efficient execution problem are approximately one minute with a solution gap[4] of less than 0.01%.

## 4.3 Result

In the case study, we ran the efficient execution model by setting a different upper bound of $CVaR$ losses across the range from -\$193,550,000 to -\$193,200,000 (i.e. revenue across the range from \$193,200,000 to \$193,550,000), increasing in increments of \$50,000, under a fixed $\alpha$ value to get an efficient frontier in the expected revenue versus $CVaR$ risk diagram. It is important to note that we treat revenue as negative losses since *CVaR* is defined on a loss distribution in maximum revenue problem.

[4] Solution gap defines a relative tolerance on the gap between the best integer objective and the object of the best node remaining. When the value |*best node-best integer* |/(*1e-10* +|*best integer*|) falls below this value, the mixed integer programming (MIP) optimization is stopped.

**Figure 4.1: Efficient Frontiers.** Plot maximum expected revenue associated to different upper bound of $CVaR$ losses across the range from -\$193,550,000 to -\$193,200,000 (i.e. revenue across the range from \$193,200,000 to \$193,550,000) in increments of \$50,000 under a fixed α value in the expected revenue versus $CVaR$ risk diagram to get an efficient frontier. Repeat the procedure for different $\alpha$ values of 0.75, 0.9, and 0.95 to get efficient frontiers under different risk preferences associated with $\alpha$ values.

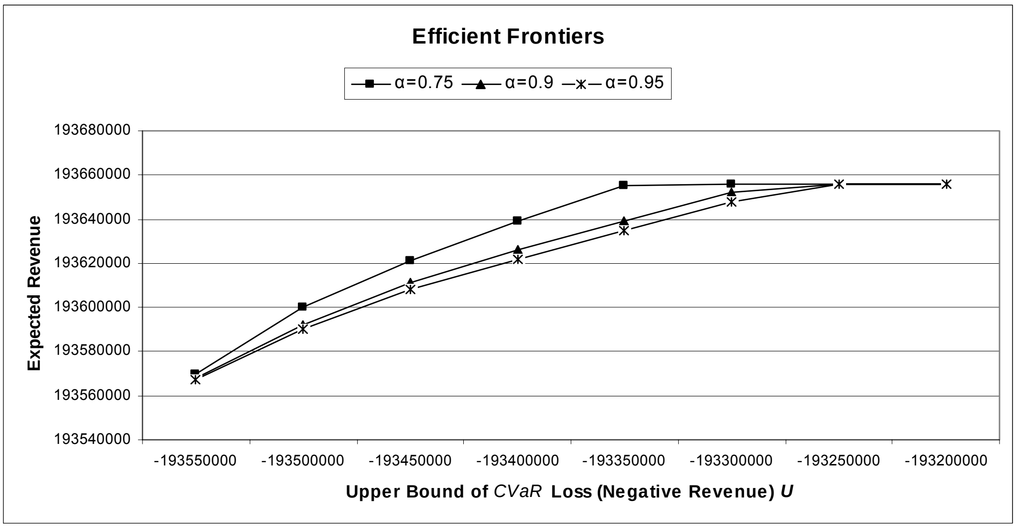



Table 4.4 summaries the solution of efficient execution under different risk preferences specified by $\alpha$ and $U$, which includes the number of mortgages sold as a whole loan, number of mortgages securitized as MBSs with a specific coupon rate ranging from 3.5% to 9.5% that increases in 0.5% increments, the number of retained mortgage servicing and released mortgage servicing, the sum of guarantee fee buy-up and buy-down amount, and the sum of excess servicing fee amount.

**Table 4.4: Summary of efficient execution solution under different risk preferences**

| | Upper Bound of *CVaR* Losses ***U*** | Total Revenue | # of Whole Loan Sale | # of mortgages pooled into MBS (with coupon rate %) 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 | # of Released Servicing | # of Retained Servicing | Sum of buy-up amount (%) | Sum of buy-down amount (%) | Sum of Excess servicing fee amount (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | -193550000 | 193570000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 429 | 134 | 59 | 5 | 0 | 845 | 0 | 45.0974 | 63.7624 | 6.59657 |
| | -193500000 | 193600000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 59 | 5 | 0 | 845 | 0 | 28.9374 | 63.6374 | 23.1316 |
| | -193450000 | 193621000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 20.041 | 63.6374 | 33.528 |
| α=0.75 | -193400000 | 193639000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 7.67415 | 63.6374 | 45.8949 |
| | -193350000 | 193655000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 21 | 192 | 428 | 134 | 55 | 10 | 0 | 845 | 0 | 0.68984 | 60.8694 | 56.1112 |
| | -193300000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 428 | 116 | 70 | 13 | 0 | 845 | 0 | 0 | 56.7976 | 66.7292 |
| | -193250000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 428 | 116 | 70 | 13 | 0 | 845 | 0 | 0 | 56.7976 | 66.7292 |
| | -193200000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 428 | 116 | 70 | 13 | 0 | 845 | 0 | 0 | 56.7976 | 66.7292 |
| | -193550000 | 193568000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 429 | 134 | 59 | 5 | 0 | 845 | 0 | 46.1211 | 63.7624 | 5.5729 |
| | -193500000 | 193592000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 59 | 5 | 0 | 845 | 0 | 33.0582 | 63.6374 | 19.0108 |
| | -193450000 | 193611000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 59 | 5 | 0 | 845 | 0 | 24.6599 | 63.6374 | 27.4091 |
| α=0.9 | -193400000 | 193626000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 16.9895 | 63.6374 | 36.5795 |
| | -193350000 | 193639000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 9.39992 | 63.6374 | 44.1691 |
| | -193300000 | 193652000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 1.03964 | 63.6374 | 52.5294 |
| | -193250000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 193 | 428 | 133 | 54 | 12 | 0 | 845 | 0 | 0 | 60.5694 | 58.501 |
| | -193200000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 428 | 116 | 70 | 13 | 0 | 845 | 0 | 0 | 56.7976 | 66.7292 |
| | -193550000 | 193567000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 429 | 134 | 59 | 5 | 0 | 845 | 0 | 46.4645 | 63.7624 | 5.22952 |
| | -193500000 | 193590000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 59 | 5 | 0 | 845 | 0 | 34.5673 | 63.6374 | 17.5017 |
| | -193450000 | 193608000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 59 | 5 | 0 | 845 | 0 | 25.959 | 63.6374 | 26.11 |
| α=0.95 | -193400000 | 193622000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 20.2216 | 63.6374 | 33.3474 |
| | -193350000 | 193635000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 12.1773 | 63.6374 | 41.3917 |
| | -193300000 | 193648000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 5.1561 | 63.6374 | 48.4129 |
| | -193250000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 193 | 428 | 133 | 54 | 12 | 0 | 845 | 0 | 0 | 60.5694 | 58.501 |
| | -193200000 | 193656000 | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 428 | 116 | 70 | 13 | 0 | 845 | 0 | 0 | 56.7976 | 66.7292 |

We then repeated the procedure for different α values of 0.75, 0.9, and 0.95 to get efficient frontiers under different risk preferences associated with $\alpha$ values. The efficient frontiers are shown in Figure 4.1, and the solutions of efficient execution under different $\alpha$ and $U$ values are listed in Table 4.4.

Figure 4.1 shows the trade-off between $CVaR$ ($\alpha$-tail risk) and expected revenue. For a fixed $\alpha$ value, when the upper bound of $CVaR$ increases, the optimal expected revenue increases in a decreasing rate. On the other hand, for a fixed upper bound of $CVaR$, a high $\alpha$ value implies high risk aversion. Therefore, the associated optimal expected revenue becomes lower.

## 4.4 Sensitivity Analysis

This section conducts a sensitivity analysis in servicing fee multipliers, mortgage prices, and MBS prices. The sensitivity analysis is performed under a $CVaR$ constraint with $\alpha$ =75% and $U$ = -\$193,400,000.

Table 4.5 shows that when all servicing fee multipliers increase by a fixed percentage, the number of mortgages sold as a whole loan decreases since lenders could get higher revenue from securitization due to the increasing servicing fee. In addition, the number of retained servicing and the amount of excess servicing fee increase due to the increasing retained servicing fee multipliers, and the number of released servicing decreases because the base servicing values, i.e. the upfront payments of released servicing, do not increase associated with retained servicing multipliers. An interesting result is that the increasing servicing fee multipliers increase (decrease) the guarantee fee buy-down (buy-up) amount, and the number of mortgages pooled into low coupon rate MBSs (4.5% and 4%) increases and the number of mortgages pooled into high coupon rate MBSs (7%, 6.5%, and 6%) decreases.

**Table 4.5: Sensitivity analysis in servicing fee multiplier**

| Servicing Fee Multipliers Increment | # of Whole Loan Sale | # of mortgages pooled into MBS (with coupon rate %) | | | | | | | | | | | | | # of Released Servicing | # of Retained Servicing | Sum of buy-up amount (%) | Sum of buy-down amount (%) | Sum of Excess servicing fee amount (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 | | | | | |
| 0% | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 7.6741 | 63.6374 | 45.8949 |
| 10% | 155 | 0 | 0 | 0 | 0 | 0 | 4 | 15 | 197 | 321 | 221 | 71 | 16 | 0 | 768 | 77 | 0 | 38.4944 | 107.926 |
| 20% | 114 | 0 | 0 | 0 | 0 | 0 | 4 | 8 | 157 | 64 | 496 | 114 | 43 | 0 | 661 | 225 | 0 | 74.0746 | 371.006 |
| 30% | 78 | 0 | 0 | 0 | 0 | 0 | 3 | 3 | 126 | 48 | 505 | 149 | 88 | 0 | 260 | 662 | 0 | 137.755 | 537.436 |
| 40% | 74 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 127 | 7 | 495 | 174 | 118 | 0 | 207 | 719 | 0 | 165.085 | 626.538 |
| 50% | 73 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 84 | 45 | 235 | 430 | 130 | 0 | 156 | 771 | 0 | 175.951 | 804.253 |

Table 4.6 shows that when all mortgage prices increase by a fixed percentage, the number of mortgages sold as a whole loan increases so that lenders can take advantage of high mortgage price. Since the number of whole loan sale mortgages increases which implies that the number of securitized mortgages decreases, the sum of buy-up and buy-down guarantee fee and excess servicing fee slightly decrease. Furthermore, the number of released servicing decreases since the number of securitized mortgage decreases.

**Table 4.6: Sensitivity analysis in mortgage price**

| Mortgage Price Increment | # of Whole Loan Sale | # of mortgages pooled into MBS (with coupon rate %) | | | | | | | | | | | | | # of Released Servicing | # of Retained Servicing | Sum of buy-up amount (%) | Sum of buy-down amount (%) | Sum of Excess servicing fee amount (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 | | | | | |
| 0% | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 7.6741 | 63.6374 | 45.8948 |
| 0.50% | 196 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 199 | 426 | 82 | 65 | 13 | 0 | 804 | 0 | 0 | 55.144 | 62.6578 |
| 1% | 277 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 198 | 406 | 36 | 52 | 12 | 0 | 723 | 0 | 0 | 48.773 | 51.4078 |
| 1.50% | 313 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 197 | 388 | 36 | 35 | 12 | 0 | 687 | 0 | 0 | 45.1942 | 49.2382 |
| 2% | 429 | 0 | 0 | 0 | 0 | 0 | 5 | 14 | 192 | 301 | 32 | 17 | 10 | 0 | 571 | 0 | 0 | 30.587 | 44.2382 |

Table 4.7 shows that when MBS price increases, the number of whole loan sale decreases, the number of securitized mortgages increases in the lower MBS coupon rate pools (4%, 4.5%, and 5%), and the number of retained servicing slightly increases.

**Table 4.7: Sensitivity analysis in MBS price**

| MBS Price Increment | # of Whole Loan Sale | # of mortgages pooled into MBS (with coupon rate %) | | | | | | | | | | | | | # of Released Servicing | # of Retained Servicing | Sum of buy-up amount (%) | Sum of buy-down amount (%) | Sum of Excess servicing fee amount (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 | | | | | |
| 0% | 155 | 0 | 0 | 0 | 0 | 0 | 5 | 30 | 183 | 428 | 135 | 56 | 8 | 0 | 845 | 0 | 7.6746 | 63.6374 | 45.8949 |
| 1% | 70 | 0 | 0 | 0 | 0 | 0 | 5 | 17 | 196 | 428 | 206 | 54 | 24 | 0 | 858 | 72 | 0.25 | 63.7476 | 62.0756 |
| 2% | 43 | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 193 | 428 | 210 | 74 | 27 | 0 | 864 | 93 | 0.25 | 65.3748 | 66.4206 |
| 3% | 29 | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 193 | 427 | 212 | 86 | 28 | 0 | 866 | 105 | 0.25 | 65.523 | 68.1438 |
| 4% | 10 | 0 | 0 | 0 | 0 | 0 | 5 | 20 | 193 | 428 | 212 | 104 | 28 | 0 | 866 | 124 | 0.25 | 66.3766 | 67.9206 |
| 5% | 1 | 0 | 0 | 0 | 0 | 0 | 5 | 21 | 192 | 428 | 212 | 113 | 27 | 1 | 867 | 132 | 0.25 | 67.6766 | 67.4706 |

# 5. Conclusion

This paper built a stochastic optimization model to perform the efficient execution analysis. The model is formulated as a mixed 0-1 linear programming problem. We first introduced mortgage securitization through MBS swap programs of GSEs and then developed a model that considers secondary marketing functionalities, including the loan-level execution for an MBS/whole loan, guarantee fee buy-up/buy-down, servicing retain/release, and excess servicing fee. Since secondary marketing involves random cash flows, lenders must balance between expected revenue and risk. We presented advantages of $CVaR$ risk measure and employed it in our model maximizing expected revenue under a $CVaR$ constraint. By solving the efficient execution problem under different risk tolerances, efficient frontiers could be found. We conducted a sensitivity analysis in parameters of expected retained servicing fee multipliers, mortgage prices, and MBS prices. The case study shows that realistic instances of the efficient execution problem can be solved in an acceptable time (approximately one minute) with CPLEX-90 solver on a PC.

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## References


Aldrich S., W. Greenberg, and B. Payner. 2001. "A Capital Markets View of Mortgage Servicing Rights." *The Journal of Fixed Income*, 11, 37–54.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath. 1999. "Coherent Measures of Risk". *Mathematical Finance*, 9, 203–228.

Downing, C., R. Stanton, and N. Wallace. 2005. "An Empirical Test of a Two-Factor Mortgage Valuation Model: How Much Do House Prices Matter?" *Real Estate Economics*, 33, 681–710.

Fabozzi, F. 2001. *The Handbook of Mortgage-Backed Securities*, 5th edition, McGraw-Hill.

Fabozzi, F. 2002. *The Handbook of Fixed-Income Securities*, 6th edition, McGraw-Hill.

Hakim, S., M. Rashidian, and E. Rosenblatt. 1999. "Measuring the Fallout Risk in the Mortgage Pipeline." *The Journal of Fixed Income*, 9, 62–75.

Hilliard, J.E., J.B. Kau, and V.C. Slawson. 1998. "Valuing Prepayment and Default in a Fixed-Rate Mortgage: A Bivariate Binomial Option Pricing Technique." *Real Estate Economics*, 26, 431–468.

Jorion, P. 2000. *Value at Risk: The New Benchmark for Managing Financial Risk*, 2nd edition, McGraw-Hill.

Kau, J.B., D.C. Keenan, W.J. Muller, III, and J.F. Epperson. 1992. "A Generalized Valuation Model for Fixed Rate Residential Mortgages." *Journal of Money, Credit, and Banking*, 24, 279–299.

Kau, J.B., D.C. Keenan, W.J. Muller, III, and J.F. Epperson. 1994. "The Value at Origination of Fixed-Rate Mortgages with Default and Prepayment." *Journal of Real Estate Finance and Economics*, 11, 5–36.

Kau, J.B. 1995. An Overview of the Option-Theoretic Pricing of Mortgages. *Journal of Housing Research*, 6, 217–244.

Lederman, J. 1997. *Handbook of Secondary Marketing*. Mortgage Bankers Association of America.

Lin, C.-C., T.-H. Chu, L.J. Prather. 2006. "Valuation of Mortgage Servicing Rights with Foreclosure Delay and Forbearance Allowed." *Review of Quantitative Finance and Accounting*, 26, 41–54.

Markowitz, H.M. 1952. "Portfolio Selection." *Journal of Finance*, 7, 77–91.

Rockafellar, R.T. and S.P. Uryasev. 2000. "Optimization of Conditional Value-at-Risk." *Journal of Risk*, 2, 21–41.

Rockafellar, R.T. and S.P. Uryasev. 2002. "Conditional Value-at-Risk for General Loss Distribution." *Journal of Banking and Finance*, 26, 1443–1471.

Schwartz, E.S. and W.N. Torous. 1989. "Prepayment and the Valuation of Mortgage-Backed Securities." *Journal of Finance*, 44, 375–392.

Stanton, R. 1995. "Rational Prepayment and the Value of Mortgage-Backed Securities." Review *of Financial Studies*, 8, 677–708.

Sugimura, T. 2004. "Valuation of Residential Mortgage-Backed Securities with Default Risk Using an Intensity-Based Approach." *Asia-Pacific Financial Markets*, 11, 185–214.

Szegö, G. 2002. "Measures of Risk." *Journal of Banking and Finance*, 26, 1253–1272.

## Appendix: Model Formulation

## Notations:

***Indices:***

$m$ = index of mortgages $(1,2,\ldots,M)$,

$M$ = total number of mortgages,

$j$ = index of mortgage groups $(1,2,\ldots,J)$,

$J$ = total number of groups,

$k$ = index of scenarios $(1,2,\ldots,K)$,

$K$ = total number of scenarios,

$c$ = index of MBS coupon rate $(1,2,\ldots,C^m)$,

$C^m$ = number of possible MBS coupon rates of mortgage *m*.

***Decision Variables:***

$$z_c^m = \begin{cases} 1, \text{ if mortgage } m \text{ is pooled into MBS with coupon rate index } c, \\ 0, \text{ otherwise,} \end{cases}$$

$$z_w^m = \begin{cases} 1, \text{ if mortgage } m \text{ is sold as a whole loan,} \\ 0, \text{ otherwise,} \end{cases}$$

$$z_{sbo}^m = \begin{cases} 1, \text{ if the servicing of mortgage } m \text{ is retained,} \\ 0, \text{ otherwise,} \end{cases}$$

$$z_{sbr}^m = \begin{cases} 1, \text{ if the servicing of mortgage } m \text{ is sold,} \\ 0, \text{ otherwise,} \end{cases}$$

$r_{gu}^m$ = guarantee fee buy-up spread of mortgage *m*,

$r_{gd}^m$ = guarantee fee buy-down spread of mortgage *m*,

$r_{ser}^m$ = retained excess servicing fee spread of mortgage *m*,

$z_k$ = real variables used in *CVaR* constraint formulation,

$\zeta$ = real variables used in *CVaR* constraint formulation.

***Input Data:***

$L^m$ = loan amount of mortgage *m*,

$P_c^{t^m}$ = price of MBS with coupon rate index *c* and maturity $t^m$,

$t^m$ = maturity of mortgage *m*,

$P_w^m$ = whole loan sale price of mortgage *m*,

$R_c$ = MBS coupon rate related to index *c*,

$K_u^m$ = guarantee fee buy-up multiplier of mortgage *m*,

$K_d^m$ = guarantee fee buy-down multiplier of mortgage *m*,

$R_n^m$ = note rate of mortgage *m*,

$R_{sb}^m$ = base servicing fee of mortgage *m*,

$R_{gb}^m$ = base guarantee fee of mortgage *m*,

$B^m$ = base servicing value of mortgage *m*,

$c_s^m$ = servicing cost of mortgage $m$,

$p_k$ = the probability of scenario $k$,

$K_{sr}^{mk}$ = retained servicing fee multiplier of mortgage $m$ under scenario $k$,

$U_{gu}^m$ = upper bound of guarantee fee buy-up spread of mortgage $m$,

$U_{gd}^m$ = upper bound of guarantee fee buy-down spread of mortgage $m$,

$U_{se}^m$ = upper bound of retained excess servicing fee of mortgage $m$,

$U_{se}^a$ = upper bound of average retained excess servicing fee of all mortgages,

$U_{se}^j$ = upper bound of average retained excess servicing fee of mortgages in group $j$,

$U$ = upper bound of $CVaR$ losses,

$\alpha$ = percentile of CVaR.

# Model Formulation:

$$
\begin{aligned}
Max \quad & \sum_{m=1}^{M}\left[ L^m \times \sum_{c=1}^{C^m}\left(P_c^{\prime m} \times z_c^m\right) + P_w^m \times z_w^m \right] \\
& \sum_{m=1}^{M}\left(L^m \times B^m \times z_{sbo}^m\right) + \sum_{m=1}^{M}\left(\sum_{k=1}^{K} p_k\left(L^m \times R_{sb}^m \times K_{sr}^{mk}\right) - c_s^m\right) z_{sbr}^m \\
& + \sum_{m=1}^{M}\sum_{k=1}^{K} p_k\left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right) \\
& + \sum_{m=1}^{M}\left(K_u^m \times L^m \times r_{gu}^m\right) - \sum_{m=1}^{M}\left(K_d^m \times L^m \times r_{gd}^m\right)
\end{aligned}
$$

| | |
|---|---|
| $\sum_{m=1}^{M}\left[ L^m \times \sum_{c=1}^{C^m}\left(P_c^{\prime m} \times z_c^m\right) + P_w^m \times z_w^m \right]$ | revenue from MBSs or whole loan sale |
| $\sum_{m=1}^{M}\left(L^m \times B^m \times z_{sbo}^m\right) + \sum_{m=1}^{M}\left(\sum_{k=1}^{K} p_k\left(L^m \times R_{sb}^m \times K_{sr}^{mk}\right) - c_s^m\right) z_{sbr}^m$ | expected revenue from the base servicing fee of securitized mortgages |
| $+\sum_{m=1}^{M}\sum_{k=1}^{K} p_k\left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right)$ | expected revenue from the excess servicing fee of securitizedmortgages |
| $+\sum_{m=1}^{M}\left(K_u^m \times L^m \times r_{gu}^m\right) - \sum_{m=1}^{M}\left(K_d^m \times L^m \times r_{gd}^m\right)$ | revenue from the guarantee fee buy-up/buy-down of securitized mortgages |

*s.t.*

| | |
|---|---|
| $\sum_{c=1}^{C^m} z_c^m + z_w^m = 1 \quad \forall m = 1,2,\ldots,M$ | mortgage $m$ must be either securitized as an MBS or sold as a whole loan |
| $z_w^m + z_{sbo}^m + z_{sbr}^m = 1 \quad \forall m = 1,2,\ldots,M$ | if mortgage $m$ is sold, there is no servicing; otherwise, servicing either be sold or be retained |
| $z_w^m + r_{gu}^m + r_{gd}^m + r_{ser}^m \le 1 \quad \forall m = 1,2,\ldots,M$ | if mortgage $m$ is sold, there is no guarantee fee and excess servicing fee |
| $\sum_{c=1}^{C^m} R_c z_c^m + r_{gu}^m - r_{gd}^m + r_{ser}^m \le R_n^m - R_{sb}^m - R_{gb}^m \quad \forall m = 1,2,\ldots,M$ | if mortgage m is securitized, note rate = MBS coupon rate + servicing fee rate + guarantee fee rate |
| $\zeta + (1-\alpha)^{-1} \sum_{k=1}^{K} p_k z_k \le U$ <br> $z_k \ge L_k - \zeta \quad \forall k = 1,\ldots,K$ <br> $z_k \ge 0, \quad \forall k = 1,\ldots,K$ <br> $L_k = -\sum_{m=1}^{M}\left[ L^m \times \sum_{c=1}^{C^m}\left(P_c^m \times z_c^m\right) + P_{whole}^m \times z_w^m \right] - \sum_{m=1}^{M}\left(L^m \times B^m \times z_{sbo}^m\right)$ <br> $-\sum_{m=1}^{M}\left(L^m \times R_{sb}^m \times K_{sr}^{mk} \times z_{sbr}^m\right) - \sum_{m=1}^{M}\left(L^m \times K_{sr}^{mk} \times r_{ser}^m\right)$ <br> $-\sum_{m=1}^{M}\left(K_u^m \times L^m \times r_{gu}^m\right) + \sum_{m=1}^{M}\left(K_d^m \times L^m \times r_{gd}^m\right)$ | $CVaR_\alpha(loss) \le U$ : <br> the expected loss (negative revenue) of the worst 1-$\alpha$ percent of senarios should be less than or equal to U |
| $\sum_{m=1}^{M} L^m \times r_{se}^m \le U_{se}^a \left(\sum_{m=1}^{M} L^m (1 - z_w^m)\right)$ | the average excess servicing fee of all securitized mortgages is restricted to an upper bound. |
| $\sum_{m \in j} L^m \times r_{se}^m \le U_{se}^j \left(\sum_{m \in j} L^m (1 - z_w^m)\right) \quad \forall j = 1,2,\ldots,J$ | the average excess servicing fee of securitized mortgages of group $j$ is restricted to an upper bound |
| $0 \le r_{ser}^m \le U_{se}^m \quad \forall m = 1,2,\ldots,M$ | for each securitized mortgage $m$, the excess servicing fee is restricted to upper bound |
| $0 \le r_{gu}^m \le U_{gu}^m \quad \forall m = 1,2,\ldots,M$ | for each securitized mortgage $m$, the guarantee fee buy-up spread is restricted to an upper bound |
| $0 \le r_{gd}^m \le U_{gd}^m \quad \forall m = 1,2,\ldots,M$ | for each securitized mortgage $m$, the guarantee fee buy-down spread is restricted to an upper bound |
| $z_c^m, z_{whole}^m, z_{sbr}^m, z_{sbo}^m \in \{0,1\} \quad \forall m = 1,2,\ldots.,C^m$ | binary constraints |
| $r_{ser}^m, r_{gu}^m, r_{gd}^m \ge 0$ | non-negative constraints |