Modifying the Richardson Arms Race Model With a Carrying Capacity

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Abstract: We endeavored to modify the Richardson Arms Race Model by introducing a carrying capacity term to each equation. These carrying capacity terms parallel the carrying capacity term introduced in a logistic growth model. As a result of these terms, new nullclines are created, thus drastically altering the resultant direction fields. We found that introducing these terms allowed us to predict the level of armament for each country upon the break of war.

I. Introduction

Lewis Fry Richardson, an English physicist, developed his arms race model after serving for France's medical corps in World War I. Deeply troubled by the events of World War I and the subsequent World War II, Richardson sought out a model that could predict large-scale military conflicts. Assuming that an arms race would be the platform from which a war was launched, he set out to model how one country's arms buildup affected the arms buildup its opponent.

II. Richardson's Model

Richardson proposed the following system of differential equations to model an arms race between nation *x* and nation *y*:

$$dx/dt = ay - mx + r \tag{1a}$$

$$dy/dt = bx - ny + s \tag{2a}$$

In this model, x is the arms expenditure for nation x at time t and y is the arms expenditure for nation y at time t. The constants a and b represent the reactions of nations x and y to the arms level of the other nation. For example, for every unit of currency nation y spends on its arms supply, then nation x increases its arms spending by a. The constants m and n are the "fatigue" terms, representing the reluctance of nations x and y to spend more of their budget on arms. To use economic terms, m and n represent the desire of nations x and y to produce butter rather than guns. The constants r and s are the hostility/peace terms. A value of r less than 0 indicates that nation x has peaceful intentions toward nation y (i.e., x will decrease if the other terms in (1a) are 0). A value of r greater than 0 indicates that nation x has hostile intentions toward nation y (i.e., x will increase if the other terms in (1a) are 0).

There are, however, several problems with Richardson's original model. First of all, the solutions to (1a) and (2a) can yield negative results. Obviously, $x, y < 0$ makes no physical sense. Also, the Richardson Model does not work for every governmental structure. For example, the Richardson Model has proven to yield accurate results for the India-Pakistan arms race but not for the Greece-Turkey arms race[1]. Lastly, as time t gets larger and the nations approach war, x and y approach infinity. Herein lies the problem with the Richardson Model for which we will attempt to account. There must exist some maximum armament level for nations x and y which cannot be exceeded.

III. Our Model

We now place budget constraints upon nations x and y by introducing carrying capacity terms to (1a) and (2a). These carrying capacity terms are of the form $(1 - x/x_{max})$ and $(1 - y/y_{max})$, similar to the carrying capacity term that transforms an exponential growth model to a logistic growth model. Our model consists of the equations:

$$dx/dt = (1 - x/x_{max}) * (ay - mx + r) \tag{1b}$$

$$dy/dt = (1 - y/y_{max}) * (bx - ny + s) \tag{2b}$$

The constants in (1b) and (2b) represent the same quantities they did in Richardson's Model. Our added terms x_{max} and y_{max} represent the maximum possible arms expenditures for nations x and y, respectively. We assume x_{max} and y_{max} to be constants. As we will see in the case study, these carrying capacity terms introduce two additional nullclines that create a "budget box" within which all solutions are contained. These nullclines cause dx/dt and dy/dt to approach zero as x and y approach their budget constraints.

It is difficult to visualize the effects our modifications have without specific values to work with. We will now consider a case study to examine the resultant outcomes of our modifications.

IV. Case Study

We assign the following values to the constants in our equations:

a	nation x's reaction coefficient	1.0
b	nation y's reaction coefficient	1.2
m	nation x's fatigue coefficient	0.9
n	nation y's fatigue coefficient	0.8
r	nation x's hostility/peace value	1.0
s	nation y's hostility/peace value	-2.0
x_{max}	nation x's maximum arms expenditure	7
y_{max}	nation y's maximum arms expenditure	9

Notice here that $r > 0$ indicates that nation x has hostile intentions toward nation y, and that $s < 0$ indicates that nation y does not desire a conflict with nation x. Richardson's Model now becomes

$$dx/dt = 1*y - 0.9*x + 1 \tag{3a}$$

$$dy/dt = 1.2*x - 0.8*y - 2 \tag{4a}$$

Our model becomes

$$dx/dt = (1 - x/7)*(1*y - 0.9*x + 1) \tag{3b}$$

$$dy/dt = (1 - y/9)*(1.2*x - 0.8*y - 2) \tag{4b}$$

As Figure 1 indicates, Richardson's Model predicts that x and y will either approach infinity or lead to disarmament as time increases. However, on Figure 2, we see that our model predicts that x and y approach an equilibrium point, E_1, or disarmament as time increases.

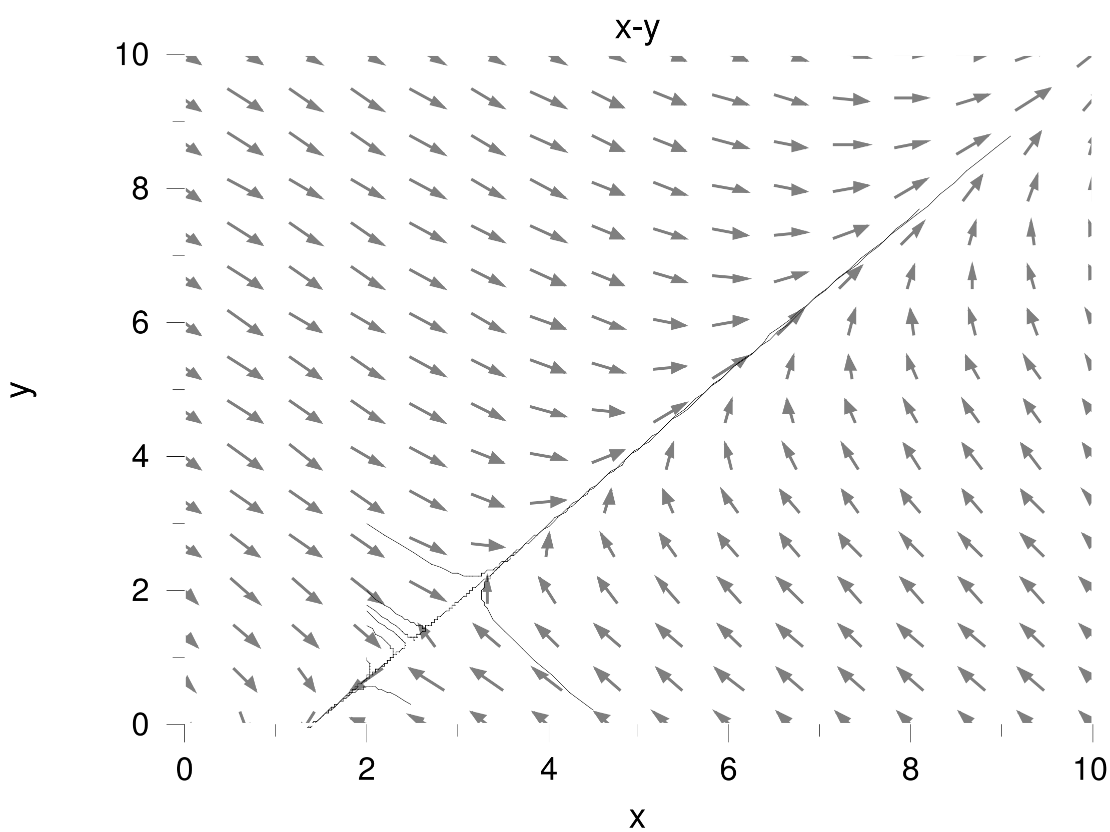

Figure 1
Results of Richardson's Model

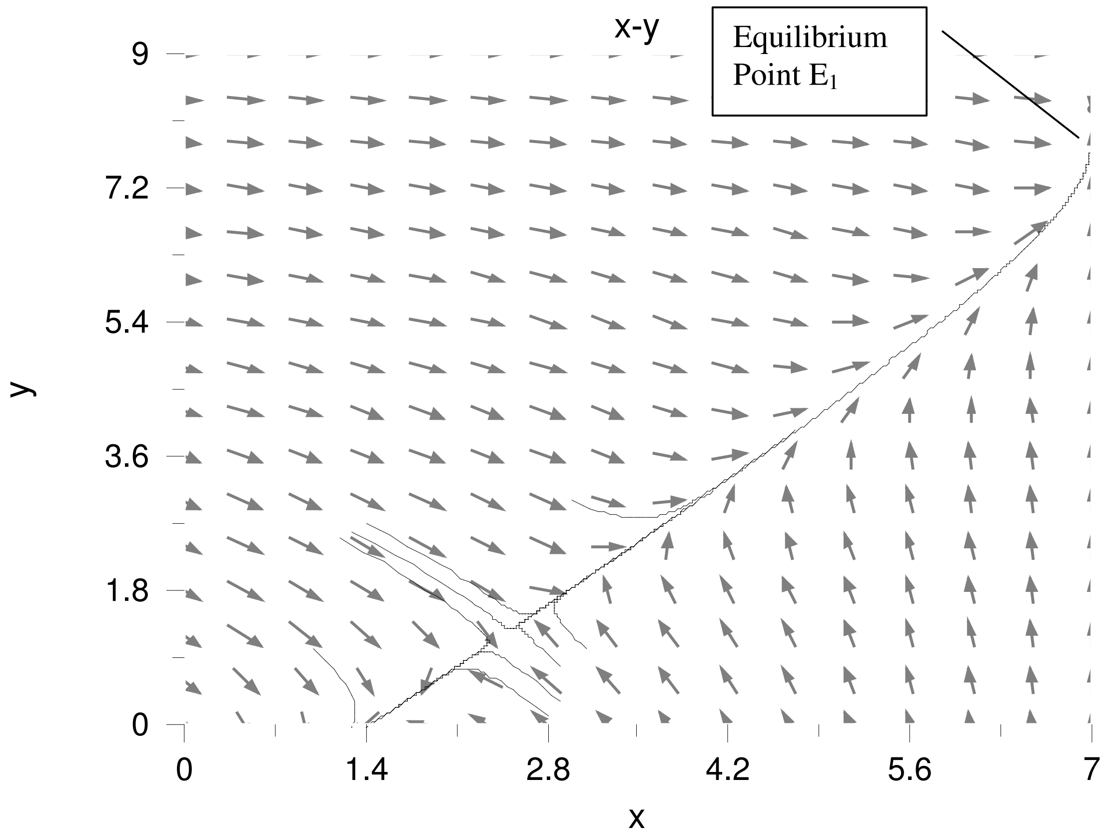

Figure 2
Results of Our Model

In order to calculate E_1, we derive the following nullclines from (3b) and (4b):

$$x^* = 7 \tag{5}$$

$$y^* = 9 \tag{6}$$

$$y^* = 1.5x^* - 2.5 \tag{7}$$

$$y^* = 0.9x^* - 1.0 \tag{8}$$

We see from Figure 3 that our equilibrium point E_1 is the intersection of nullclines (5) and (7), which gives us the point (7,8). Next we will determine the stability of this equilibrium point by linearization.

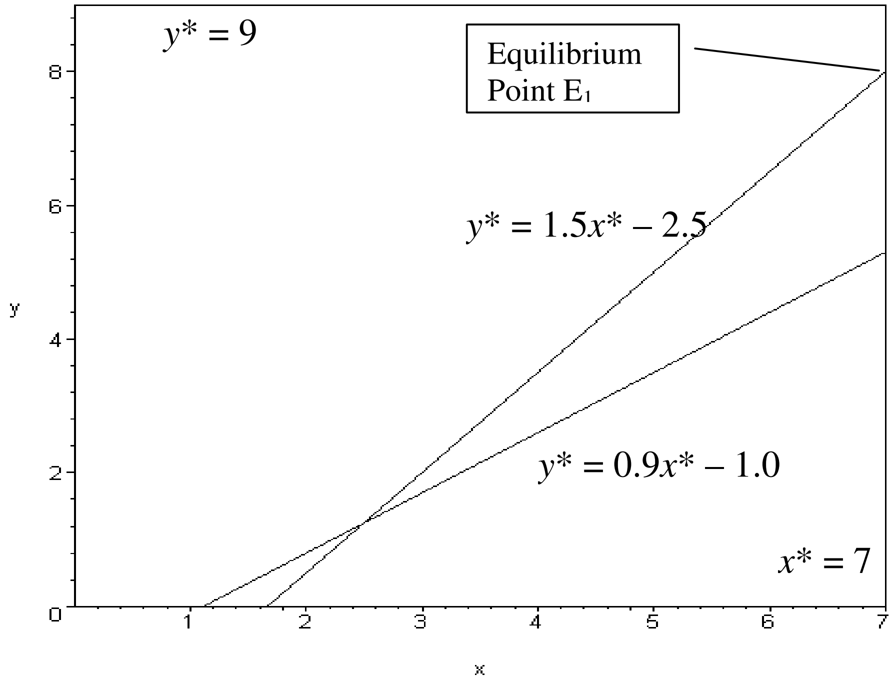

Figure 3
Nullclines

V. Linearization Around (7,8)

In order to analyze the stability of the system around (7,8), we will consider the Jacobian matrix for the system. Letting $F(x,y) = dx/dt$ and $G(x,y) = dy/dt$, the Jacobian is:

$$J := \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \tag{9}$$

For our model, the Jacobian is:

$$\mathrm{J}(x, y) := \begin{bmatrix} -\frac{a\,y - m\,x + r}{xmax} - \left(1 - \frac{x}{xmax}\right) m & \left(1 - \frac{x}{xmax}\right) a \\ \left(1 - \frac{y}{ymax}\right) b & -\frac{b\,x - n\,y + s}{ymax} - \left(1 - \frac{y}{ymax}\right) n \end{bmatrix} \tag{10}$$

Substituting our case study constants into (10), and evaluating at (7,8), we have:

$$J(7, 8) := \begin{bmatrix} -0.3857142857 & 0 \\ 0.1333333333 & -0.01111111111 \end{bmatrix} \tag{11}$$

Computing the eigenvalues (λ) for the equilibrium point (7,8) yields:

$$\lambda = -0.0111, -0.3857 \tag{12}$$

Since the real parts of both eigenvalues are less than zero, we conclude that E_1 is a stable equilibrium point.

VI. Conclusions

As our case study has shown, because of the added terms of constraint, our arms race model has an equilibrium point not found in Richardson's Model. Whereas Richardson's Model can only predict whether two nations will go to war or disarm (x and y approach infinity or zero), our model can predict at what values x and y will level out upon the outbreak of war. This would be of immense importance to nations x and y throughout an arms race, giving them an idea of where the escalating tensions are headed.

Our model also gives nations x and y two additional parameters to alter in their favor. For example, if nation x could increase its maximum arms expenditure—perhaps by acquiring more resources—it could shift the final equilibrium point in its favor. Also, if nation x could decrease nation y's maximum arms expenditure, it could better ensure its own victory.

One of the main disadvantages to our model is that we cannot find an exact solution to our model, since it is nonlinear, while Richardson's Model is linear, making an exact solution possible. Also, in creating our model, we assumed x_{max} and y_{max} to be constants. They may be constant for short periods of time, however they will most likely be variant over time. Replacing x_{max} and y_{max} with functions of time would eliminate this problem, however it would make analysis of the model much more difficult.

References

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4. M. D. Ward, "Modeling the USA - USSR arms race", <http://www.unibw-hamburg.de/WWEB/math/uebe/Lexikon/D/Defense-Bsp01.pdf>